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EDITORIAL COMMENT AND NEWS NOTES

This Issue of the Journal

Because of the statewide interest of teachers, supervisors, and administrators in the Elementary Mathematics In-Service Laboratory held at San Diego State College, June 19-23, 1961, this issue of the *Journal* is devoted to the material presented by leaders in mathematics education who participated in the Laboratory. The five-day session was the result of a co-operative effort of the Bureau of Elementary Education and the Bureau of National Defense Education Act Administration of the California State Department of Education and San Diego State College to provide an opportunity for educators to secure an answer to their questions about what is "new" in mathematics for the elementary school.

The program for the conference included an hour-and-a-half lecture (8:00-9:30 a.m.) on the following topics by Margaret Willerding, Associate Professor of Mathematics, San Diego State College:

Monday—*Bases Other Than Ten.* Study of the Hindu-Arabic System of Notation with emphasis on bases other than ten

Tuesday—*Properties of the Arithmetic Numbers.* Study of the properties of the integers and rational numbers, number line

Wednesday—*Mathematical Systems.* Introduction of operations other than addition and multiplication, modular arithmetic and its properties

Thursday—*Sets and Geometric Concepts.* Introduction to set theory, Venn diagrams, union and intersection of sets, the universal set, using set theory in the language of geometry

Friday—*Number Theory.* Study of prime numbers, factors, divisibility

The program provided opportunity for participants to meet in small groups from 10:30-11:30 a.m. and from 1:00-3:30 p.m. with staff leadership as follows: San Diego State College: Richard Madden, Director, Marguerite Brydegaard, Margaret Willerding, Francis Ballentine, James Inskeep.

California State Department of Education: Helen Heffernan, Chief, Bureau of Elementary Education; Consultants, Bernard J. Lonsdale, Mrs. Afton D. Nance, Mrs. Ester Nelson, Ruth Overfield, Robert Reynolds, Robert Shute.

Consultants provided by the Bureau of Elementary Education and the Bureau of National Defense Education Act Administration: G. Don Alkire, Fresno State College, Emily V. Baker, Office of the San Bernar-

dino County Superintendent of Schools; Mrs. Leslie S. Beatty, Chula Vista Public Schools; Ferrell G. Ingham, Imperial School, Downey; L. Clark Lay, Orange County State College; C. Richard Purdy, Alameda County State College.

The material presented by members of the State Advisory Committee on Mathematics has been included in this issue. Space in this issue did not permit summarization of questions raised by the participants and informal discussion regarding the application of principles to specific grade levels. However, all consultants emphasized the following:

1. The importance of a precise mathematical vocabulary so children can describe mathematical concepts with accuracy.
2. New ideas can be inserted into present practice without waiting for the development of new mathematics curriculum or instructional materials.
3. In introducing new ideas into the mathematics curriculum, the needs of children and the nature of learning should not be disregarded.
4. The new approach to mathematics involves wider use of discovery techniques.

The Laboratory was skillfully directed by Richard Madden, whose introductory statements, questions, and summaries did much to clarify concepts and deepen the insight of participants. Robert Shute, Consultant, Elementary Education (NDEA), not only served as a resource consultant in mathematics education, but handled all details of registration, distribution of materials, exhibit and the countless details essential to a successful professional meeting.

More than 250 persons registered for the conference and urged that similar conferences be held regionally over the State. If funds become available, mathematics laboratory sessions including opportunity for observation and critical evaluation of mathematics teaching with emphasis on methods of instruction at various grade levels will be organized for the summer of 1962.

Reading Attainment in Elementary Schools

As elementary school teachers, supervisors, and administrators note the new state textbooks available in reading at the beginning of the school year, they are conscious of the importance of effective utilization of this generous array of modern reading materials to increase the achievement of elementary school children.

Gates' recent study of reading attainment in elementary schools is of timely interest to all who share responsibility for the reading program offered by the public schools.[1]

[1] Arthur I. Gates, *Reading Attainment in Elementary Schools 1957 and 1937*. New York: Bureau of Publications Teachers College, Columbia University, 1961.

> . . . certain critics of education have declared . . . that today's youngsters at all educational levels read less well than the youth of a generation ago. The evidence presented in support of this declaration is typically negligible, usually a subjective opinion based on casual observation of a limited population, and thus likely to be wholly unreliable. In the studies in which objective tests of reading ability have been given to two generations of pupils in the same community, it has been found without exception that children in schools during the past decade have surpassed those of two or more decades earlier, in some cases by substantial amounts.

In the study reported a comparison is made of the reading atainment of today's children with those measured 20 years earlier using the four batteries of the Gates Reading Tests.

Dr. Gates presents the basic data to support his contention with a thoroughly acceptable sampling of representative school populations.

At a time when California schools are initiating use of two new basic reading series, a careful examination of Dr. Gates' study will indicate (1) the importance of determining the present reading attainment of children now in elementary schools; (2) comparison of these data with data on children attending school in the district one or two or three decades ago; and (3) projection of studies to determine the effectiveness of the materials and methods which will be in use during the decade immediately ahead.

As Dr. Gates points out in making such studies, it will be necessary to make allowances for such differences as the following:

1. The general intelligence or scholastic aptitude of the pupils
2. The average chronological age at the various grade levels
3. The average number of years spent in school after entering the first grade
4. The number of children repeating each grade
5. The number dropping out of school at each grade level
6. The nature and relative difficulty in the old and new . . . reading tests
7. The equivalence of scores on the old and the new tests
8. Other factors such as the economic and educational status of the parents, the educational climate, and types of recreational and other incentives operating in the home and the community

This formidable list of variables should not act as a deterrent to any school district in pursuing similar studies designed to produce objective evidence of the reading attainment of California elementary school children. The people of the state have made a great investment in providing new material reinforced by the best methods of teaching

reading currently supported by research. Rarely in educational history is educational material subjected to such a crucial test as will be made in the decade ahead with more than two million children enrolled in the schools of California.

WHAT IS THE "NEW" MATHEMATICS? [1]

MARGARET E. WILLERDING, *Associate Professor of Mathematics, San Diego State College*

In answering the question "What is the 'New' Mathematics?" for teachers, supervisors, and administrators engaged in elementary education, kindergarten through grade eight, attention will be directed to the following topics: Systems of numeration; properties of the rational numbers; mathematical systems; sets; and number theory.

SYSTEMS OF NUMERATION

Symbols

A symbol is something that represents something else. Usually the symbol has little resemblance to the thing it represents.

Using symbols is not new to you. For example, when you see the symbol you think *cat*. This symbol actually resembles the object it represents. When you see the word "cat" you also think of cat, although the symbol "cat" does not resemble the actual physical object.

We use many different types of symbols in life. For example, you may smile to indicate that you are pleased; a person may wear a pin to indicate his membership in a certain organization.

In mathematics, we use many symbols. You are familiar with some of them: $+$, $\times$, $-$, $\div$, $\frac{1}{2}$, and the like.

Even though we understand what symbols represent, we may still have difficulty because we confuse the symbol with the thing it represents. This is especially true in mathematics.

Often we confuse a mark on paper such as "7" with the idea it represents. We may wish to refer to the symbol which represents a particular thing or idea. In mathematics, when we refer to the symbol itself we place quotation marks about the symbol. When we refer to the thing or idea itself, we omit the quotation marks. For example: when I write "15" the "1" means 10 and the "5" means 5.

There are many symbols used in mathematics which you should learn. Certain of these symbols, along with their meanings and uses, appear on the next page.

[1] During the Elementary Mathematics In-Service Laboratory at San Diego State College, June 19-23, 1961, Dr. Willerding lectured for an hour and a half each day on mathematics content in order to provide a background of material likely to prove of greatest interest and value to elementary school pupils and teachers. The material is presented in complete form to assist persons who were not able to attend the Laboratory to help answer their questions about what is "new" in mathematics.

Symbol	*Meaning*	*Uses*
$=$	"is equal to"	$2+2=4$
$\neq$	"is not equal to"	$3+7\neq 7$
$>$	"is greater than"	$2>1$
$<$	"is less than"	$2<7$
$\cdot$	"times"	$2\cdot 3=6$
2^4	$2\times 2\times 2\times 2$	$5^2=5\cdot 5=25$

Numbers and Numerals

In mathematics we often tend to confuse numbers with the symbols which represent them. We call the symbols which represent numbers numerals. Numerals are not numbers; numerals only represent numbers. For example: "2," "⅓," "0.75," "3 + 2" are numerals.

If "2" is not a number, what is it? The number 2 is an idea associated with certain sets of objects. For example, 2 is the number associated with each of the following sets:

It is impossible to define number. The idea, number, is one of the most abstract concepts that man has conceived. Mathematicians say that "numbers" are undefined.

Systems of Numeration with Bases Other Than Ten

The first thing is to distinguish between a *system of numeration* and a *number system*. A number system is a broader concept than a system of numeration. A number system deals with numbers and their properties irrespective of the particular notation used to denote the numbers. A system of numeration, on the other hand, is concerned with the symbols used for writing the numbers and is not concerned with the properties of these numbers.

The principal elements of a place system of numeration are (1) the digits; (2) the base; and (3) the principle of position or place value.

In building a system of numeration, the first step is to create symbols to indicate the numbers. These may be tally marks such as the Egyptians used, or they may be the symbols 0, 1, 2, 3, . . ., 9, such as we use today. In either case these symbols represent a 1-1 correspondence between individual, distinct objects.

The second step is to invent symbols to represent a collection of a specific number of objects rather than individual objects. In the Egyptian system of numeration this is illustrated by the scroll, , which represents one hundred objects; in our system this is illustrated by the symbol 100 which represents one hundred objects.

The next step is the introduction of the idea of place value and in our system the zero, along with its symbol, to symbolize the empty place.

Perhaps the most important idea is the principal of exponential or expanded notation. This means the positional arrangement into an ordered sequence. The sequence consists of consecutive powers of the base of the particular system.

For example, in our decimal system

$$384 = 3 \cdot 100 + 8 \cdot 10 + 4 = 3 \cdot 10^2 + 8 \cdot 10^1 + 4 \cdot 10^0$$
$$138{,}645 = 1 \cdot 100{,}000 + 3 \cdot 10{,}000 + 8 \cdot 1{,}000 + 6 \cdot 100 + 4 \cdot 10 + 5 = 1 \cdot 10^5 + 3 \cdot 10^4 + 8 \cdot 10^3 + 6 \cdot 10^2 + 4 \cdot 10^1 + 5 \cdot 10^0$$

Before looking at systems of numeration with bases other than ten, we may review briefly the Hindu-Arabic System with base ten.

The symbols used for the digits are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. Since the base of the decimal system is ten, ten digits are used. In any system of numeration with the same characteristics as the decimal system as many digits are used as the base. In other words, if we build a system of numeration with base eight we need eight digits; for the binary system (base two) we need two digits; for the duodecimal system (base twelve) we need twelve digits.

We use these symbols, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 in combinations to make any number no matter how large or how small. Since the base of the decimal system is ten, we use groups of ten. We may review briefly the principle of exponential notation expressing the digits of a numeral as successive powers of the base; that is

$$325 = 3 \cdot 100 + 2 \cdot 10 + 5 = 3 \cdot 10^2 + 2 \cdot 10^1 + 5 \cdot 10^0$$
$$1462 = 1 \cdot 1000 + 4 \cdot 100 + 6 \cdot 10 + 2 = 1 \cdot 10^3 + 4 \cdot 10^2 + 6 \cdot 10^1 + 2 \cdot 10^0$$

This is called the expanded or exponential notation of a number. The exponential notation of 7,864 is $7 \cdot 10^3 + 8 \cdot 10^2 + 6 \cdot 10^1 + 4 \cdot 10^0$

Now we may use the same general characteristics as the decimal system and build a system of notation with base eight.

Since the base of the system is eight we need eight digits. We may use the familiar digits of the base ten system. Then we need digits 0, 1, 2, 3, 4, 5, 6, 7.

In the decimal system (base ten) each succeeding place, reading from right to left has a value ten times as great as the preceding place. In the base eight system each succeeding place reading from right to left will have a value *eight* times as great as the preceding place. Thus in the base eight system the first place on the right will be the units place, the next place the eights, the next the 8^2 or 64's place and so on, each place being eight times as great as the succeeding place. Thus in the scale of eight, the numeral 43 means

$$43 = 4 \cdot 8^1 + 3 \cdot 8^0,$$

that is, 43 in base eight is the same as 35 in the base ten.

Similarly,

$$524 = 5 \cdot 8^2 + 2 \cdot 8^1 + 4 \cdot 8^0 \text{ (340, base ten)}$$
$$1037 = 1 \cdot 8^3 + 0 \cdot 8^2 + 3 \cdot 8^1 + 7 \cdot 8^0 \text{ (543, base ten)}$$

From now on a subscript will be used to denote the base in which a numeral is written. For example, 32_{five} means that the numeral is written in a base five system.

It is quite simple to convert a numeral in base eight to an equal numeral in base ten. All that needs to be done is to write the numeral in exponential notation and find its value.

$$174_{eight} = 1 \cdot 8^2 + 7 \cdot 8^1 + 4 \cdot 8^0 = 1 \cdot 64 + 7 \cdot 8 + 4 = 64 + 56 + 4 = 124_{ten}$$
$$36_{eight} = 3 \cdot 8^1 + 6 \cdot 8^0 = 24 + 6 = 30_{ten}$$

To change a numeral in base ten to an equal numeral in base eight is equally as simple. You have a basket full of pennies and you wanted to find how much money you have, first you would put the pennies into stacks of ten pennies each. Then you would put these stacks of ten pennies each into sets of ten stacks of ten pennies each. Then you would put these sets of ten stacks into sets of ten, and so on. Precisely the same thing is done when a numeral is converted from base ten to base eight.

To change 94_{ten} to an equal numeral in base eight, divide by eight to find how many eight's:

$$8 \overline{)\,94}$$
$$\quad 11 \quad 6$$

This shows we have eleven stacks with eight in each stack and six left over. In other words, we have 6 units.

Now these eleven stacks are put into sets of eight:

$$8 \overline{)\,11}$$
$$\quad 1 \quad 3$$

We have one set of eight stacks, and three stacks of eight left over. Hence $94_{ten} = 136_{eight}$

As another illustration we may change 231_{ten} to an equal numeral in base eight, as follows:

$$8 \overline{)\,231}$$
$$8 \overline{)\,28} \quad \text{7 or seven } 8^0\text{'s}$$
$$8 \overline{)\,3} \quad \text{4 or four } 8^1\text{'s}$$
$$0 \quad \text{3 or three } 8^2\text{'s}$$

Thus $231_{ten} = 347_{eight}$

The successive digits used to express the number in base eight are the series of remainders taken in the reverse order from that in which

they were obtained. The divisions are repeated until a quotient zero is reached. Each remainder must be less than the base since the base is being used as the divisor. If a remainder of zero is obtained, the corresponding power is used zero times.

It is interesting and illuminating to construct addition and multiplication tables for numbers in different bases.

We may construct the addition and multiplication tables for base eight, as follows:

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7	10
2	2	3	4	5	6	7	10	11
3	3	4	5	6	7	10	11	12
4	4	5	6	7	10	11	12	13
5	5	6	7	10	11	12	13	14
6	6	7	10	11	12	13	14	15
7	7	10	11	12	13	14	15	16

x	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	4	6	10	12	14	16
3	0	3	6	11	14	17	22	25
4	0	4	10	14	20	24	30	34
5	0	5	12	17	24	31	36	43
6	0	6	14	22	30	36	44	52
7	0	7	16	25	34	43	52	61

Since the binary system is so important in the computer field, we may spend a short time studying it. In this system only two symbols,

0 and 1, are needed. As noted before, any number may be expressed as the sum of successive powers of the base. Thus:

$$14_{ten} = 1110 = 1 \cdot 2^3 + 1 \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0$$
$$25_{ten} = 11001 = 1 \cdot 2^4 + 1 \cdot 2^3 + 0 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0$$

The first twenty numerals in the binary system are 1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111, 10000, 10001, 10010, 10011, 10100.

The two digits 0 and 1 are the same in the binary system as in our decimal system. To express two, we cannot write "2" since this symbol does not exist; so we move to the next column to the left and write 10. Now we can use the units' place again to count one more, 11 or "three." To get "four" we write 100. The places from right to left in the binary system are: units, two's, four's, eight's, sixteen's, thirty-two's, and so on.

The duodecimal system is the base twelve system. Some people recommend that this system should replace our decimal system. Since the duodecimal system has base twelve, we need twelve digits. If we use our ordinary symbols, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, we need two others to represent ten and eleven. We cannot use 10 to represent ten because in the base twelve system 10 means one dozen. If we use the symbol x (read dek) for ten and E (read el) for eleven, the symbol $TE2_{(twelve)} = 10 \cdot 12^2 + 11 \cdot 12^1 + 2 \cdot 12^0 = 1574_{ten}$

Properties of the Rational Numbers

The numbers represented by the sequence of numerals

1, 2, 3, . . ., 99, 100, 101, . . .

are called the counting numbers, since they are the only numbers needed to count how many. Zero is not a counting number because in counting there is no element before the first element. Through experience we observe certain properties of the counting numbers. The first property is that the set of counting numbers has a beginning but it has no end, that is there is no last counting number. We call such an unending set an *infinite set.*

We also know that we can "add" counting numbers. Addition is thought of as putting two sets together to form one set. When we add 3 and 2 we obtain 5. Actually we do not get "more" when we add. All we do is regard the two sets as one set.

It is always possible to add two counting numbers. Intuitively we know that when we add two counting numbers the result is also a counting number. Moreover for any pair of counting numbers, there exists only one counting number which is their sum. We call this property the *closure property.* Mathematically this is expressed by saying that *the set of counting numbers is closed under addition.*

We observe, too, that addition is commutative. This means that we can commute or reverse the order in which two numbers are added without affecting their sum. Thus:

$2 + 3 = 3 + 2$
$7 + 9 = 9 + 7$
$a + b = b + a$ where a and b are any counting numbers.

Finally, addition is associative, which means, for example:

$$3 + (5 + 7) = (3 + 5) + 7$$
$$3 + 12 = 8 + 7$$
$$15 = 15.$$

A parenthesis placed about the indicated sum of two numerals emphasizes the fact that those two numerals are to be regarded as a symbol for one number. The word "associate" suggests that when adding three numbers we may put a parenthesis around the first pair or the second pair as we wish; the resulting sum is the same whichever way they are associated. In other words if *a, b,* and *c* are any counting numbers the associative property says

$$(a + b) + c = a + (b + c).$$

Intuitively, we know that multiplication with counting numbers is always possible. When any two counting numbers are multiplied the result is a unique counting number called the product. Mathematically this means the set of *counting numbers is closed under multiplication.*

We observe, again through experience, that multiplication is commutative:

$3 \times 4 = 4 \times 3$
$8 \times 7 = 7 \times 8$
$a \times b = b \times a$ where a and b are counting numbers.

Multiplication, like addition is also associative. For example

$(3 \times 4) \times 5 = 3 \times (4 \times 5)$
$(7 \times 2) \times 8 = 7 \times (2 \times 8)$
$(a \times b) \times c = a \times (b \times c)$ where a, b, and c are counting numbers.

Still another property is exhibited by counting numbers. This property, called the *distributive* property, ties together the operations of addition and multiplication. For example:

$$3 \times (4 + 6) = (3 \times 4) + (3 \times 6)$$
$$3 \times (10) = 12 + 18$$
$$30 = 30$$
$9 \times (5 + 2) = (9 \times 5) + (9 \times 2)$
$a \times (b + c) = (a \times b) + (a \times c)$ where a, b, and c are counting numbers.

Before examining further properties of the counting numbers, let us look at the idea of equality.

This relation of equality has certain properties.

1. If *a* is a counting number, $a = a$.
2. If *a* and *b* are counting numbers and $a = b$, then $b = a$.
3. If *a*, *b*, and *c* are counting numbers and if $a = b$ and $b = c$, then $a = c$.
4. If *a*, *b*, *c*, and *d* are counting numbers and if $a = b$ and $c = d$, then $a + c = b + d$.
5. If *a*, *b*, *c*, *d* are counting numbers and if $a = b$ and $c = d$, then $ac = bd$.

In the set of natural or counting numbers there exists a unique number designated by one (1) with the property that $a \times 1 = 1 \times a = a$, where a is any natural number. Notice

$4 \times 1 = 1 \times 4 = 4$
$9 \times 1 = 1 \times 9 = 9$
$7 \times 1 = 1 \times 7 = 7$

The number 1 is called the *identity element for multiplication.*

In dealing with natural numbers every number has a successor. This successor is found by adding 1 to the number. The successor of 1 is $1 + 1 = 2$; the successor of 2 is $2 + 1 = 3$, and so on.

If we picture the counting numbers on a number line we can see some other important properties of the counting numbers, namely the property of order.

Notice that between any two numbers only one of three relations can exist:

(1) $a = b$
(2) $a < b$
(3) $a > b$.

The larger the number the farther to the right it is on the number line; if $a < b$ this means that *b* is to the right of *a* on the number line.

We also can see that if $a + c = b + c$, then $a = b$; and if $ac = bc$ then $a = b$.

Although the set of counting numbers contains an identity for multiplication it contains no identity for addition. This means when you have only the set of counting numbers you cannot find a number, let's call it z, such that

$$a + z = z + a = a$$

for every natural number *a*. But you know a number that has such a property. What is it? Zero, of course. If zero is included with the set

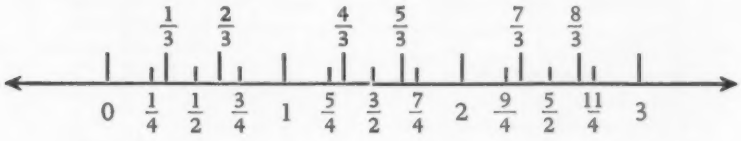
of counting numbers a set is obtained which is called the set of whole numbers. This set of whole numbers has all the properties (with one exception) of the counting numbers with the addition of an identity (zero) for addition.

The exception is in the assumption, if *a*, *b*, and *c* and *d* are counting numbers and if $a = b$ and $c = d$, $ac = bd$. Here we must say $c \neq 0$.

Zero has an important characteristic which we shall mention but not prove:

$$a \cdot 0 = 0 \cdot a = 0 \text{ where a is any whole number.}$$

If we look again at the number line, we can see that we have determined points on the line corresponding to the whole numbers. The number ½ we associate with the point midway between 0 and 1. By laying off this segment of length one-half over and over again we determine the points on the line corresponding to ½, 3/2, 5/2, ... Next by using a length which is one-third of the unit length and measuring this length successively to the right of zero, we locate the points ⅓, ⅔, 4/3, 5/3, ... Similarly, we locate the points to the right of 0 on the line corresponding to fractions having denominators 4, 5, 6, 7 . . . .



By this natural process we associate with each rational number a point on the line. Just one point on the line is associated with each rational number.

Will we use up the supply of points in this process of assigning numbers to points? No, even though between any two rational numbers no matter how close together they are there is another, there are still points on the line that do not represent a rational number. These points are associated with numbers called *irrational numbers.*

This set of rational numbers have all the properties of the whole numbers plus another called the inverse for multiplication. Notice that for every number, except zero, there is another number such that the product of these two numbers is one:

$2 \cdot ½ = 1$ $\qquad$ $5 \cdot ⅕ = 1$

$3 \cdot ⅓ = 1$ $\qquad$ $a \cdot 1/a = 1$ where *a* is a rational number.

Now we have discussed the whole set of numbers used in elementary arithmetic. We have used only half of the number line in associating

the rational numbers with points on a line. All the numbers are to the right of 0. Are there numbers associated with points to the left of 0? As before, we use the interval from 0 to 1 as the unit of measure, and locate points equally placed along the line to the left. The first of these we label -1, the second -2 and so on. Proceeding as before, we can find the negatives of all the rational numbers. Thus the negative of 3/2 is the same distance from 0 on the left as 3/2 is on the right. Including the negatives of the irrationals we now have the set of numbers called the *real numbers*.

This new set of numbers must satisfy all the properties of the arithmetic numbers. From experience we know that $(a) + (-a) = 0$. In other words $-a$ is the additive inverse of a.

Because these negative numbers must satisfy the assumptions governing the positive numbers, it is easy to show that a negative number times a positive number yields a negative number and that a negative number times a negative number yields a positive number. Demonstrations of these two assumptions are as follows:

Starting with the assumption	$3 \cdot 0 = 0$
Substituting for zero	$3[2 + (-2)] = 0$
Using the Distributive Law	$3 \cdot 2 + 3(-2) = 0$
Substituting for $(3-2)$	$6 + 3(-2) = 0$
Since	$6 + (-6) = 0$
It follows that	$3(-2) = -6$
Similarly starting with	$(-2)(0) = 0$
Substituting for zero	$(-2)[3 + (-3)] = 0$
Using the Distributive Law	$(-2)(3) + (-2)(-3) = 0$
Substituting for $(-2)(3)$	$-6 + (-2)(-3) = 0$
Since	$-6 + (+6) = 0$
It follows that	$(-2)(-3) = +6$

Mathematical Systems

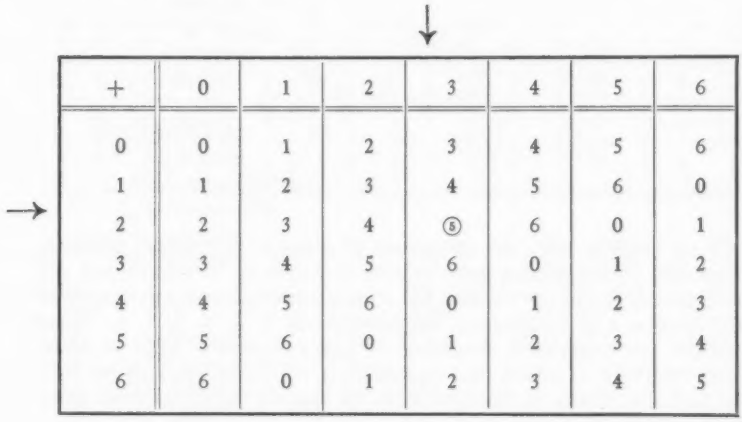
The sketch below represents the face of a seven-minute clock. Zero is the starting point and, also, the end-point of a rotation of the hand.

With the model we might start at 0 and move two spaces, thus the hand will be at 2. If we follow this by moving one more space, we would stop at 3. Thus $2 + 1$ gives 3. Now suppose we start at 0 and move 5 spaces and follow this by a move of 3 spaces. Then $5 + 3$ gives 1. A common way to write this is

$$5 + 3 \equiv 1 \pmod 7.$$

This is read "5 plus 3 is congruent to 1, modulo 7". Modulo 7 is abbreviated mod 7. The (mod 7) means that there are only seven numerals, 0, 1, 2, 3, 4, 5, 6, on the face of the clock. The plus, +, sign means what has just been described. This is a new kind of addition.

Now we may construct the addition table for our system as shown in the following illustration.



+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	(5)	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

We read this table by following across horizontally from any entry in the left column, for example 2, to the position below some entry in the top column, such as 3. The entry in this position in the table is taken as the result of combining the element in the left column with the element in the top column, in that order:

$$2 + 3 \equiv 5 \pmod 7$$

Now we may define what we mean by clock multiplication.

Suppose we start at zero and make three moves of two spaces each. We say

$$3 \times 2 \equiv 6 \pmod 7$$

then

$$4 \times 3$$

means starting at zero and making four moves of three spaces each. Then

$$4 \times 3 \equiv 5 \pmod 7$$

We have now defined multiplication in this new mathematical system. In constructing the mathematics table we develop one such as follows:

X	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

We are familiar with the operations of ordinary arithmetic, addition, subtraction, multiplication and division of numbers. We have just discussed two different operations. We made a table for the new type of multiplication and addition of the numbers 0, 1, 2, 3, 4, 5, 6. These operations are completely described by the two tables. That is, there are no numbers to which the operation is applied except those indicated and the results of the operations as applied on all pairs of these numbers are given.

So far the only operations we have had have been called addition or multiplication. We may call operations we define by any name we wish and give them any symbols we desire. For example:

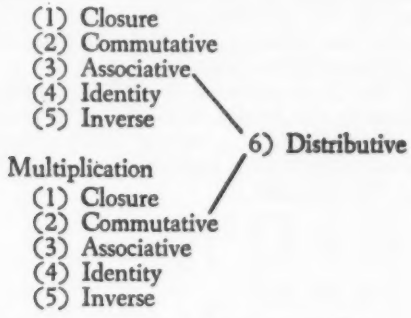
$\odot$	0	1	2	3
0	0	1	2	3
1	2	3	4	5
2	4	5	6	7
3	6	7	8	9

All of operations which have been defined are binary operations because they are applied to two elements to get a third element. So far, the elements have been numbers but they need not be.

We note that modular addition and multiplication are commutative in the sense that the same answer is obtained by use of the commutative principle. However, from an operational standpoint, modular multiplication is not communtative since for example the expression *two* 3's has a different operational meaning than *three* 2's.

We may note how many of the properties of rational numbers our modular addition and multiplication satisfy.

First, we may consider addition. Is there closure? That is, when two elements of the system are added do I get a unique element of the system?



Now, we may check the operation ⊙.

(1) Closure	No
(2) Commutative	No
(3) Associative	No
(4) Identity	Yes
(5) Inverse	No

THE OBSESSION OF MRS. RODENT [2]

OR

LAZINESS IS THE MOTHER OF THE INVENTION OF A MATHEMATICAL SYSTEM

Most women move furniture around occasionally in an effort to give a room "that added something." But most women, after having experimented with the same unsuccessful arrangement several times, will give it up. The wife of our hero, Mr. Rodent, was different. She could not resist the hope that if she tried a given arrangement just once more she might be able to make it work out.

[2] Carl H. Denbow, "The Furniture Mover's Story," *Foundations of Mathematics*. New York: Harper & Bros., 1959.

Imagine then an ordinary living room containing among other things a sofa, a radio, and a chair. (From now on S will denote the sofa, C, the chair, and R, the radio.) At the end of each hard day's work as a cheese nibbler, Mr. Rodent comes home to find that his wife has decided to try a new arrangement of these three objects. Her "orders of the day" are explicit and written in a note that looks like this

$$\begin{bmatrix} R & S & C \\ C & R & S \end{bmatrix}$$

Mr. Rodent knows only too well what this means. "Put the chair where the radio now is, put the radio where the sofa now is and put the sofa where the chair now is." Mr. Rodent has received so many of these little billets-doux that he has named them. He calls this one G because it was first handed to him on green paper.

Mr. Rodent early learned that he is expected to replace the objects named in the top row by those in the bottom row and not vice versa. The top symbols represent "before" and the bottom "after." He also noted that there are several messages which look different but which contain the same instructions. For example:

$$\begin{bmatrix} S & R & C \\ R & C & S \end{bmatrix}$$

contains exactly the same instructions as the message he calls G. In fact, when he tries to tabulate all possible messages, he finds only six basically different ones. These are:

$$E = \begin{bmatrix} R & S & C \\ R & S & C \end{bmatrix} \quad C = \begin{bmatrix} R & S & C \\ S & R & C \end{bmatrix} \quad G = \begin{bmatrix} R & S & C \\ C & R & S \end{bmatrix}$$

$$A = \begin{bmatrix} R & S & C \\ R & C & S \end{bmatrix} \quad J = \begin{bmatrix} R & S & C \\ S & C & R \end{bmatrix} \quad B = \begin{bmatrix} R & S & C \\ C & S & R \end{bmatrix}$$

He looks at E (meaning easy) with special longing; it is the message which requires no work.

Now the names, G, A, B, C, E, and J are only names, and you may be inclined to think they are not very important. A name is only a name. It moves no furniture. But don't assume this agnostic attitude.

A set of language agreements is an important part of establishing a new system of thought, and apt naming of a system frequently clarifies thinking and leads to new insights.

As Mr. Rodent now recalls it, his interest in the implications of this little game first occurred to him when his wife's aunt became ill. One night Mr. Rodent came home and found the inevitable message (this time it was A) with the note which said, "I am spending the night with poor Aunt Susan. Your dinner is in the refrigerator." Mr. Rodent was very hot and decided to have a beer and read the paper before he moved the furniture, and, human nature being what it is, when the next evening came the furniture was still unmoved and another message (B) had arrived. Instead of actually moving the furniture twice, he sat down and wrote this on a piece of paper.

$$A = \begin{bmatrix} R & S & C \\ R & C & S \end{bmatrix} \quad B = \begin{bmatrix} R & S & C \\ C & S & R \end{bmatrix}$$

~~R S C~~
~~R C S~~
C R S

$$A + B = \begin{bmatrix} R & S & C \\ C & R & S \end{bmatrix}$$

He found that move A followed by move B resulted in the same arrangement as move G.

This excited him so that he sat down and made the following table:

SECOND MOVE

	E	A	C	J	G	B
E	E	A	C	J	G	B
A	A	E	J	C	B	G
C	C	G	E	B	A	J
J	J	B	A	G	E	C
G	G	C	B	E	J	A
B	B	J	G	A	C	E

He was quite pleased with himself, because he realized that now he could leave his furniture moving chores pile up until his wife was due home, at which time he could add up the accumulated notes and by one single manipulation produce the same furniture arrangement as the entire set of manipulations.

But alas, this wonderful table was the cause of a terrible dispute between Mr. Rodent and his wife. His wife left him message B on Monday night and left for Aunt Susan's; on Tuesday night she sent over message A. Now Mr. Rodent recalled that he had already proved that $A + B = G$: so he assumed that $B + A$ must also result in G. He had just completed manipulation G on the furniture when his wife arrived and asked him, in icy tones, just what he thought he was doing. Flushed and unhappy, he worked out $B + A$ and found alas that $B + A = J$! And so in this horrible way Mr. Rodent found that his addition of moves was not commutative!

This discovery so upset Mr. Rodent that he studied his addition table every free moment that he had. He soon saw that he was dealing with a kind of mathematics in which the things to be added were not *numbers* but *operations*. He also discovered that his addition could be used for any situation in which the operations were similar to his furniture moving, whether the objects rearranged were pieces of furniture, or moons of Jupiter, or carbon atoms in a molecule. In other words, he was concerned with the study of certain operations rather than the nature of the objects being operated upon.

Being a pretty bright fellow (even though kind of a mouse!), Mr. Rodent discovered that the addition of these operations (or elements) has some feature in common with the addition of numbers. For example E plays a role similar to that of zero in addition.

$$\begin{array}{ll} 2 + 0 = 2 & A + E = A \\ 3 + 0 = 3 & B + E = B \\ 5 + 0 = 5 & J + E = J \end{array}$$

From then on he called E the "zero element."

Mr. Rodent's next discovery came as a result of Aunt Susan's taking a turn for the worse. (Individual welfare frequently has to be sacrificed for the advancement of science.) There came a day when Mr. Rodent found himself with not two but *three* accumulated messages, so he had to perform the computation $B + J + G$. He noted at once that his table permitted him to add only two elements at a time, and that this gave him a choice of procedure. If he added $B + J$ first he would obtain A, and if then he performed $A + G$ he obtained B. By using parentheses he summarized these results like this:

$$(B + J) + G = A + G = B.$$

Since this is exactly the grouping which the problem required (after all, if he had been doing his job properly he would at the end of the

second day, have performed B + J, and he would then on the third day have performed G), therefore B was undoubtedly the right answer.

Mr. Rodent was now so overcome by scientific curiosity that he performed

$$B + (J + G) = B + E = B$$

and found that, in this case at least

$$(B + J) + G = B + (J + E).$$

He became so excited that he performed all the possible combinations he could and found that this is always true. In other words if x, y, and z are elements in Mr. Rodent's table

$$(x + y) + z = x + (y + z).$$

This is the same as the associative law of addition of numbers.

As Mr. Rodent admired his table, he noted, with longing that there were several pairs which gave E as a sum; so that if his luck were good his wife might give him two successive messages which totaled E and he wouldn't have to do any work at all. For example:

$$G + J = E$$
$$A + A = E$$

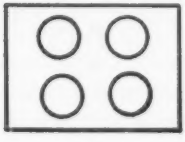
In other words J is the move which undoes move G and leaves the furniture as if no manipulation had been performed at all. For this reason Mr. Rodent called J the inverse of G. He studied and found that each element in his table has an inverse. Can you find them?

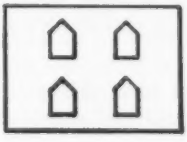
The more Mr. Rodent studied his table the more he was impressed by the differences between addition of these elements and addition of integers. He realized that $G + J = E$ is something like $7 + (-7) = 0$; in other words, J can be regarded as something like the negative of G. But then what about $A + A = E$? Here is an element that is its own negative. If you add it to itself you get zero. No integer does this. He saw that he must be very careful to avoid carrying preconceptions from one system of mathematics to another. And so must you!

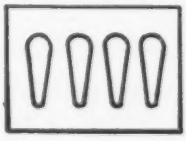
And the moral of this is that all mathematics, (even advanced mathematics because this is), need not be dull!

Sets

What is a set? A *set* is simply a collection of things. The idea of set was met in your first formal experiences with arithmetic. You were probably shown various sets of objects, such as the pictures that follow:







You were then asked to name the number associated with each of these sets. Young children learn to associate numbers with sets of objects. The concept of addition is also introduced by means of sets. For example, a set of two dogs and a set of three dogs is combined to form a new set. The number 5 is associated with the new set.

Sets are important in all levels of mathematics. The science of sets tells us how to combine sets and how to compare sets to determine relationships. Solving number sentences, drawing graphs, describing geometric figures all become simpler by using the ideas and language of sets.

A set may be a collection of objects, numbers, persons, figures or other sets. The objects making up a set are said to *belong to the set*. These items which belong to the set are called *elements* or *members* of the set.

In studying sets we will use small letters of the alphabet to represent the elements of a set instead of the pictures used in primary arithmetic. For example, if we wish to talk about a set with five elements we will represent the elements by the letters a, b, c, d, and e. Braces, { }, are used to enclose the members of a set. Thus the set consisting of a, b, and c is written

$\{a, b, c\}$

We use capital letters as names of sets. Thus

$A = \{a, b, c\}$

When each member of a set is tabulated, as in set A, we say we have *listed* its members. Another way to designate the members of a set is to describe them.

Thus $R = \{\text{Alaska, Hawaii}\}$ may be described as R $= \{$the 49th and the 50th states of the United States of America$\}$.

Equal and Equivalent Sets

Whether or not you have ever formally been introduced to the concept of sets, you have no doubt compared two sets. When you ask yourself if there are more men than women at a party you are comparing two sets—the set of women with the set of men at the party.

One way to compare sets, the one you used when you compared the two sets above, is to compare the number of elements in one set with the number of elements in another set. If two sets have the same number of elements, they are said to be *equivalent* sets. Sets A and B below are equivalent sets:

$A = \{1, 2, 3, 4\}$
$B = \{2, 4, 6, 8\}$

If two sets have exactly the same elements they are called *equal sets.*

$C = \{2, 3, 1, 4\}$

Set C is equal to set A. It does not make any difference whether the elements of the two sets are in the same order or not. Just as a committee of Joy, Ann, and Tom is the same committee as one consisting of Ann, Tom, and Joy, a set having elements a, b, and c, is the same as a set having elements, b, a, and c. When two sets such as sets A and C have the same elements we write

$A = C.$

Empty Set

It is useful to have a set that contains no elements. In the set consisting of all the three dollar bills in circulation in the United States, we know that this set contains no elements. A set that contains no elements is called the *empty* or the *null set.* The empty set is represented by the symbol $\emptyset$. It is also shown like this { }. We shall use the symbol $\emptyset$ to represent the empty set.

The empty set is similar to zero in our number system. Other examples of empty sets follow:

1. The set of all rectangles with 5 sides
2. The set of odd numbers that are divisible by 2
3. The set of whole numbers greater than 3 and less than 4

One-to-one Correspondence

Two sets may be compared by matching the elements of one set with the elements of another set. The sets $A = \{1, 2, 3, 4\}$ and $B = \{5, 6, 7, 8\}$ are not equal. The elements of A are not the same as the elements of B, but both sets have the same number of elements. Every element of A can be matched with an element of B. Notice that

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ 5 & 6 & 7 & 8 \end{array}$$

every element of A is matched with only one element of B, and every element of B is matched with only one element of A. This matching of elements is called a *one-to-one correspondence.*

The idea of one-to-one correspondence is not limited to the pairing of elements of two sets. If a professor enters his classroom and finds that every student has a chair and there are no vacant chairs, he knows immediately that there is a one-to-one correspondence between the students in the class and the chairs in the room.

Since two equivalent sets have the same number of elements; the elements of two equivalent sets may be put into one-to-one correspondence.

Finite and Infinite Sets

Some sets have so many elements that it would be tedious to list them all; for example, the set of natural numbers less than 5,000. We use three dots to show that there are some elements left out

$$\{1, 2, 3, \ldots 5000\}$$

The three dots followed by the numeral 5,000 means "and so on in the same manner up to 5,000."

Sometimes there is no end to the number of elements in a set. For example, the set of natural numbers; no matter how many natural numbers you make there are always more. We say there are infinitely many elements in this set and call it an *infinite set.* The number of elements in an infinite set is always greater than any number you can count. The sets below are examples of infinite sets:

1. Set of points on a line
2. Set of odd numbers
3. Set of even numbers
4. Set of prime numbers

When a set is not infinite, it is called a *finite set.* You can always count the elements of a finite set and come to an end. Examples of finite sets are:

1. Set of natural numbers less than 10
2. Set of grains of sand on Mission Beach

Subsets

Any set B, all of whose elements are also elements of another set A, is called a *subset* of A. For example, if A is the set of natural numbers

$$A = \{1, 2, 3, \ldots\}$$

and B is the set of even numbers

$$B = \{2, 4, 6, \ldots\}$$

every element of B is also an element of A because every even number is also a natural number. The set B is called a subset of A. We express "B is a subset of A" by writing $B \subset A$.

This idea of subsets is not restricted to mathematics. For example, suppose the Mathematics Club has members Weber, Jones, Whipple, Allen, Jordan, Summers, and Adkins, and the Social Committee of the Mathematics Club has members Whipple, Allen and Jordan. Then

$$M = \{\text{Weber, Jones, Whipple, Summers, Allen, Jordan, Adkins}\}$$
$$S = \{\text{Whipple, Allen, Jordan}\}$$

Then $S \subset M$.

The set of mathematics majors is a subset of the set of students of a college.

If every freshman is required to take a class in orientation, then the set of freshmen and the set of freshmen taking orientation are equal sets. In other words, if F is the set of freshmen and O is the set of freshmen taking orientation,

$$O = F \text{ and } O \subset F$$

Thus we see that every set is a subset of itself.

The empty set, $\emptyset$, is also considered as a subset of every set. The set itself and the empty set are called *improper subsets*, all other subsets of a set are called *proper subsets*.

If $A \subset B$ and $B \subset A$, then $A = B$, for if $A \subset B$, then every element of A is an element A, and if $B \subset A$, then every element of B is an element of A. But if every A is a B and every B is an A, then $A = B$.

How many subsets can be formed from a set of n elements? The order of the elements in a set is unimportant; we wish to find how many combinations of the elements we can get without regard to order. The set itself and the null set is included in every list of subsets.

We may start with the very simple set $A_1 = \{1\}$. Its subsets are $\{1\}$ and $\emptyset$. Now take the set $A_2 = \{1, 2\}$. Its subsets will include both the subsets of A_1 plus $\{1, 2\}$ and $\{2\}$. We see that there are twice as many subsets when a set has two elements as when it has one element.

Now if we take the subsets of $A_3 = \{1, 2, 3\}$, they will include all the subsets of A_2 plus $\{3\}$, $\{1, 3\}$, $\{2, 3\}$, and $\{1, 2, 3\}$. We see that again we double the number of subsets by adding one element to the set.

Examine the following tabulation:

Set	*Subsets*	*Number of Subsets*
$\{1\}$	$\emptyset$ $\{1\}$	$2 = 2^1$
$\{1, 2\}$	$\emptyset$, $\{1\}$, $\{2\}$, $\{1, 2\}$	$2 \cdot 2 = 4 = 2^2$
$\{1, 2, 3\}$	$\emptyset$, $\{1\}$, $\{2\}$, $\{3\}$, $\{1, 2\}$ $\{1, 3\}$, $\{2, 3\}$, $\{1, 2, 3\}$	$2 \cdot 2 \cdot 2 = 8 = 2^3$
$\{1, 2, 3, 4\}$	$\emptyset$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{1, 2\}$ $\{1, 3\}$, $\{1, 4\}$, $\{2, 3\}$, $\{2, 4\}$ $\{3, 4\}$, $\{1, 2, 3\}$, $\{1, 2, 4\}$ $\{1, 3, 4\}$, $\{2, 3, 4\}$ $\{1, 2, 3, 4\}$	$2 \cdot 2 \cdot 2 \cdot 2 = 16 = 2^4$

How many subsets can be made from a set of five elements? Probably you have discovered that the rule for the number of subsets for a set with n elements is 2^n. For example, if a set has four elements we can get $2^4 = 16$ different subsets (see table). If a set has seven elements, we can get $2^7 = 128$ subsets.

The Universal Set

The set of all the elements under consideration in a particular problem is called the *universal set* or the *universe*. If we select committees

from the members of a club, the club members are the elements of the universe and the committees are subsets of the universe.

Sometimes it is not clear what the universal set is unless we describe it specifically. Suppose the members of a set are football players. The universal set may be the football players of a particular college, football players from colleges in the middlewest, or all football players.

We use the symbol U to designate the universal set. For example, if

$$U = \{1, 2, 3, 4, 5\}$$

then the set $\{1, 2, 3\}$ is a subset of U. If U is the set of natural numbers the set $\{1, 2, 3, 4, 5\}$ is a subset.

Venn Diagrams

To understand the properties of sets, we use a schematic representation called a *Venn Diagram*. A rectangle represents the universal set. The elements of U are represented as points on or inside the rectangle. The size of the rectangle has nothing to do with the number of elements of U.

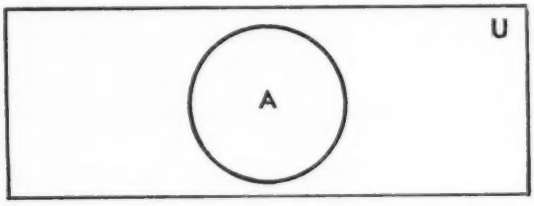


Figure 1

Venn Diagrams are used to picture the relationships between sets rather than their comparative size. A subset of U is usually represented by a circle inside of U. In the diagram above, suppose U represents the set of natural numbers and A represents the set of all odd numbers. Notice that both U and A have infinitely many elements. On the other hand, suppose that $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $A = \{2, 4, 6, 8\}$. The diagram would still picture the sets properly.

A subset of A is presented by another circle within Circle A.

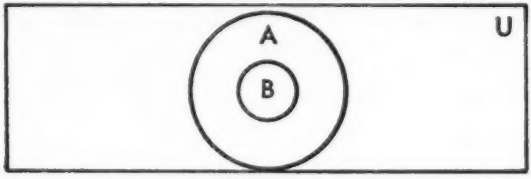


Figure 2

If U = {natural numbers}, A = {even natural numbers}, and B = {2, 4, 6, 8, 10} the relationship of A, B, and U is shown above.

If you consider the ways any two circles can be related, you will know how any two sets can be related. The three ways that two circles may be related are shown in the three diagrams that follow.

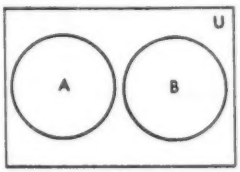


Figure 3

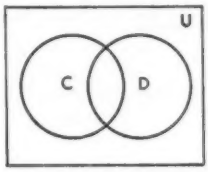


Figure 4

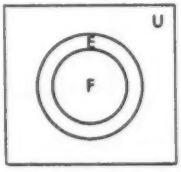


Figure 5

Two circles can be separate as in Figure 3. In other words, cirlces A and B have no points in common. The sets represented by A and B have no elements in common and are called *disjoint sets.* An example of two disjoint sets are

A — set of all odd numbers
B — set of all even numbers.

Another example of disjoint sets is

X — set of all cats
Y — set of all dogs.

Two circles can have some points in common and intersect as in Figure 4. The sets represented by C and D have some elements in common and are called *intersecting sets.*
An example of intersecting sets is

C = {1, 2, 3, 4, 5, 6}
D = {2, 4, 6, 8, 10, 12}.

Another example of intersecting sets is

P = set of all college students taking a physics class
Q = set of all college students taking a mathematics class.

One circle can be completely inside another circle as in Figure 5. This means that all the points of the inner circle are points of the outer circle. This means that all the elements of the set represented by F are members of the set represented by circle E. In other words, F is a subset of E.

When more than two sets are related, we can still show their relationship with a Venn Diagram as represented in Figure 6.

The sets U = natural numbers
A = {1, 2, 3, 4, 5, 6}
B = {3, 4, 6, 8, 10, 18, 26}
C = {1, 2, 6, 10, 18}

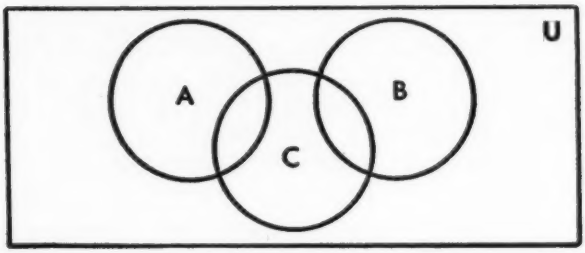


Figure 6

Union and Intersection

In arithmetic we have the operations of addition, subtraction, multiplication and division. We use these operations to combine numbers in arithmetic. The two main operations for combining sets are called *union* and *intersection.*

Suppose we consider the following two sets:

$A = \{1, 2, 3, 4, 5, 6, 7, 8\}$
$B = \{2, 4, 6, 8, 10, 12\}$

Suppose we wish to find the set which consists of the elements which are in A or B or both. This set consisting of the elements which are in A or B or both is

$\{1, 2, 3, 4, 5, 6, 7, 8, 10, 12\}$.

Then

$A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8, 10, 12\}$.

We read $A \cup B$, "A union B".

Any two sets have a union. We can show the union of two sets on a Venn Diagram. The set $A \cup B$ is shown on the next page in Figure 7 in cross hatching.

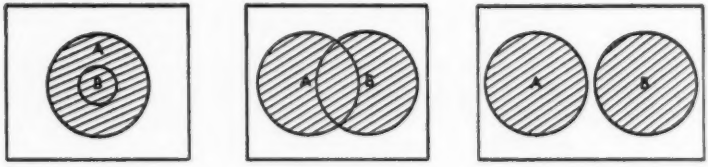


Figure 7

The operation of union may be extended to include more than two sets. For example, if we consider the three sets A, B, and C, the union of A and B is a set $(A \cup B)$ which may be combined with the set C to form the set $(A \cup B) \cup C$. The crosshatched area in Figure 8 shows $(A \cup B) \cup C$.

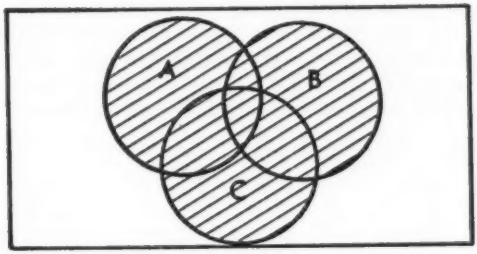


Figure 8

We may consider sets

$X = \{2, 4, 6, 8, 10\}$
$Y = \{2, 3, 4, 5, 6, 7\}$.

We may wish to find the set which consists of the elements which are in both X and Y. This set consists of elements which are the common elements of X and Y:

$\{2, 4, 6\}$.

This set is called the *intersection* of X and Y. The symbol for intersection is $\cap$.
Then

$X \cap Y = \{2, 4, 6\}$.

We read $X \cap Y$ as "X intersection Y."

Any two sets have an intersection. The intersection of two sets may be the empty set. For example, if

$O = \{\text{odd numbers}\}$
$E = \{\text{even numbers}\}$
Then $O \cap E = \varnothing$.

We can show the intersection of two sets on a Venn Diagram. The sets $X \cap Y$ is shown in Figure 9 in *cross hatching*.

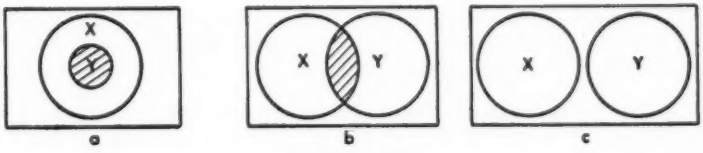


Figure 9

When two sets are disjoint as X and Y in Figure 9-c, the intersection is the empty set.

The operation of intersection may also be extended to include more than two sets. For example, if we consider the three sets X, Y and Z, the intersection of X and Y is a set of X Y which may be combined with set Z to form the set $(X \cap Y) \cap Z$. The *crosshatched* area in Figure 10 shows $X \cap Y$. The *crosshatched* area in Figure 11 shows $(X \cap Y) \cap Z$.

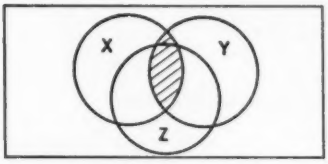


Figure 10

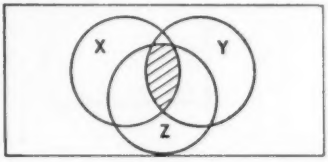


Figure 11

If we consider sets

$X = \{1, 2, 3, 4, 5\}$
$Y = \{2, 4, 6, 8\}$
$Z = \{1, 3, 5, 4\}$

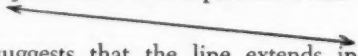
Then $X \cap Y = \{2, 4\}$, and $(X \cap Y) \cap Z = \{4\}$.

Geometry

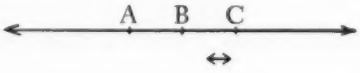
The study of geometry of space is important in this age of space travel. In geometry the concept of set is most important. In geometry we meet the terms point, line, plane, and space. We may begin with a point. What is a point? We may think of a point as being a location in space. We represent it with a dot like a period (.). But a point has no shape or size or weight. A point is really an idea. Nobody has ever seen a true point. We may think of the tip of a fine needle as a point and yet the point of a needle contains infinitely many mathematical points. In fact, mathematicians do not even give a definition for a point. They say a point is an undefined term and discuss the properties of a point.

We think of a line as an infinite set of points. The term "line" means "straight line." All lines in geometry are understood to be straight. A line is suggested by the edge of a ruler or the intersection of two walls of a room.

A geometric line extends without limit in each of two directions. It does not stop at a point. A line is represented like this



The arrowhead suggests that the line extends indefinitely in both directions. If we wish to call attention to several points on a line, we do it like this:



$\overleftrightarrow{}$

Another symbol for the above line is AB. When we consider only part of a line between two particular points, we call that part of the line a *line segment*. The part of the line between A and B is a line segment. The points A and B are end points of the *line segment*.

Any point on a line separates the line into two *half lines*. Half lines are sets of points. The point A on the line above determines three sets of points: (1) the set of points to the left of A; (2) the set of points to the right of A; and (3) the set containing only point A. If we join A to one of the half-lines we call the half-line with A adjoined a *ray*. A ray is really the union of the set of points in a half-line and the point that separates a line into two half lines. The intersection of two rays on a line determined by a point on the line is the point itself. The intersection of two half-lines determined by a point on the line is the empty set.

A flat surface such as the floor suggests the idea of a plane in mathematics. Like a line, a plane is thought of as being unlimited in extent. If you begin at a starting point on a plane and follow paths on the plane in all possible directions, these paths will remain in the plane but have no endings. A plane, then, would have no boundaries.

We think of a plane as containing infinitely many points and infinitely many lines, we represent a plane by a portion of it, as

Now let us discover some properties of points, lines and planes.

If you hold a string in your two hands and stretch it taut, in any position where the string is straight it marks a portion of a line. If you hold your hands in one position, how many positions are there for the taut string? You have just discovered a basic property of geometry:

Property 1: In any two different points in space there is exactly one line.

Suppose we represent two points on the board. If you draw the line determined by these two points, will it lie in the plane of the board? Will it ever get out of the plane of the board? We have discovered another property of geometry:

Property 2: If a line contains two different points of a plane it lies in the plane.

We think of space as being a set of points. Space, then, is an infinite set of points. It contains infinitely many lines and infinitely many planes.

We may ask ourselves some questions. How many planes contain a specified pair of points? A book with its pages spread apart answers

this question. Through two points there are infinitely many planes.

We may think of three points, as the bottom of the legs of a three-legged stool. How many planes contain these three points?

A three-legged stool will always stand solidly against the floor, while a four-legged stool does not always do this. This should suggest:

Property 3: Any three points not on the same line are in only one plane.

It would be interesting to discover other geometric properties, but since time is short we will concentrate on terms of modern geometry. The ideas are probably familiar to you, but the terminology may be new.

New Terms

1. Half Plane

2. Triangle

(a) a triangle is a figure composed of three points, not all on a single line, and the three line segments joining them in pairs.

(b) a triangle separates a plane into three sets of points, (1) the set on the triangle; (2) the set inside the triangle; (3) the set outside the triangle.

3. Angle

(a) A set of points on two rays with a common end point.

(b) There is a set of points inside or interior of the angle and a set of points outside or exterior of the angle.

4. Curve

(a) A curve is a set of points which can be represented by a pencil drawing made without lifting the pencil off the paper.

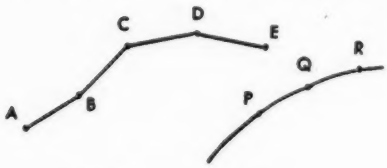


Figure 1 Figure 2

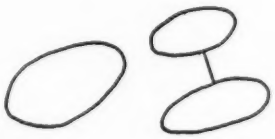


Figure 3 Figure 4

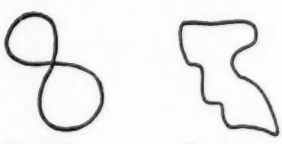


Figure 5 Figure 6

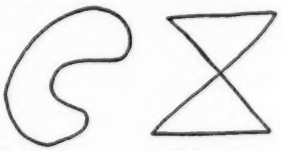


Figure 7 Figure 8

(b) Broken line curve—union of several line segments (Figure 1).

(c) Simple closed curve—a curve represented by a figure drawn in the following manner—the drawing starts and stops at the same point and no point is touched twice. (Figures 3, 6, 7).

(d) Figures 4, 5, 8 cannot be drawn without lifting pencil from paper, so they are not curves

5 Region—interior of a simple closed curve.

Number Theory

When you learned to add and subtract common fractions by finding the least common denominator, changing to like fractions, and reducing, the process was not difficult so long as the numbers involved were not too large and you knew your multiplication tables. However, when the problems involved numbers beyond the memorized multiplication tables it was not always easy to find the least common denominator or to know for certain that fractional answer, such as $^{8}4\!/_{51}$, was reduced as far as possible. The "trial and error" system often used takes too long and is uncertain. A method can be learned whereby you can find a common denominator and be certain it is the least common denominator. A similar method will enable you to reduce common fractions to their lowest terms and leave you in no doubt about the answer. Whether adding or subtracting fractions or reducing fractions to lowest terms, for a quicker, easier, more accurate solution, you will need to learn about prime numbers and factoring.

The Sieve of Eratosthenes

About 2200 years ago a Greek mathematician and scientist named Eratosthenes devised a pattern of numbers which is now referred to as the Sieve of Eratosthenes. This pattern locates all of the numbers which are multiples of certain special numbers called *prime numbers.* A prime number is a whole number greater than one which is divisible only by itself and 1.

For example, 2 is a prime because the only numbers that divide 2 are 1 and 2; 7 is a prime because the only numbers that divide 7 are 1 and 7; 4 is not a prime because 4 is divisible by 1, 2, and 4. A number that is not a prime is called a *composite number.* The first few primes are 2, 3, 5, 7, 11, 13 and 17. What is the next prime? With the exception of 2, are there any other even primes? Why?

Eratosthenes devised a method of finding all of the prime numbers less than some given number. His method sieved out all the numbers that are not primes and left only the primes. We may try using this method to find all the primes less than 50.

We write in a table all the numbers greater than 1 but less than 50, as follows

	2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~
11	~~12~~	13	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~
~~21~~	~~22~~	23	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	29	~~30~~
31	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	37	~~38~~	~~39~~	~~40~~
41	~~42~~	43	~~44~~	~~45~~	~~46~~	47	~~48~~	~~49~~	

We leave the prime, 2, and mark out every second number after 2. All the numbers which we have marked out are divisible by 2 and hence are not primes. The next number is 3 which is a prime. Now we mark out every third number after 3. All the numbers which we have marked out are divisible by 3 and hence are not prime numbers. The next number not marked out is 5 which is a prime. Now we mark out every fifth number which is a multiple of 5 and hence not a prime. The next number not marked out is 7, a prime. We mark out every seventh number. Continuing in this manner we can find all the prime numbers less than 50. They are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47.

In the sieve some numbers such as 45 are marked out more than once. This happens because 45 is divisible by more than one prime number. In fact, it is divisible by both 3 and 5.

Factoring

We may consider the composite number 144. We see that $144 = 2 \times 72 = 3 \times 48 = 4 \times 36 = 8 \times 18 = 9 \times 16 = 12 \times 12 = 16 \times 9 = 18 \times 8 = 36 \times 4 = 48 \times 3 = 72 \times 2$. We say that 144 is a multiple of 2, 3, 4, 6, 8, 9, etc., and that 2, 3, 4, 6 etc., are *factors* of 144. When two or more numbers are multiplied together to get a product, the numbers themselves are called *factors* of the product.

If we write 18 as a product of factors we have

$$18 = 1 \times 18 = 2 \times 9 = 3 \times 6 = 2 \times 3 \times 3$$

When speaking of the factors of a number, the number 1 and the number itself are always factors. When asked for the factors of a given number, all the factors should be given. For example, the factors of 18 are 1, 2, 3, 6, 9, 18.

When we factored 18 into $2 \times 3 \times 3$ we noticed that all of the factors were prime numbers. Whenever it is desirable to factor a number into prime factors, we say that you *factor the number completely*. Every number larger than 1 that is not a prime can be written as a product of one and only one set of primes.

For example 20 can be either of the following ways

$$20 = 4 \times 5 = 2 \times 2 \times 5$$
$$20 = 2 \times 10 = 2 \times 2 \times 5.$$

Notice that the prime factors of 20 are 2, 2, and 5.

When a large number is to be factored completely the most efficient method is by using division. If we want to factor 288 into its prime factors. The smallest prime is 2; so we would first divide by 2. The following example shows the prime factors or prime divisors of 288.

$$28 = 2 \times 2 \times 2 \times 2 \times 2 \times 3 \times 3 = 2^5 \times 3^2$$

$2 \mid \underline{288}$

$$
\begin{array}{r|l}
2 & 144 \\
\hline
2 & 72 \\
\hline
2 & 36 \\
\hline
2 & 18 \\
\hline
3 & 9 \\
\hline
 & 3
\end{array}
$$

The following examples show that you can use the division method to find the prime factors of a given number.

$396 = 2^2 \times 3^2 \times 11$

$$
\begin{array}{r|l}
2 & 396 \\
\hline
2 & 198 \\
\hline
3 & 99 \\
\hline
3 & 33 \\
\hline
 & 11
\end{array}
$$

$1025 = 5^2 \times 41$

$$
\begin{array}{r|l}
5 & 1025 \\
\hline
5 & 205 \\
\hline
 & 41
\end{array}
$$

Factoring Large Numbers

When numbers are relatively small, little difficulty is involved in finding other numbers that will divide them evenly. When a number divides another number without a remainder we say that the first number is a *divisor* of the second. For example, 5 is a divisor of 325, 2 is a divisor of 168 and 10 is a divisor of 1840.

To find the factors of a number we can always guess and divide, but it is much easier if you can tell from looking at a number whether or not it is divisible by small numbers. If you look at the Sieve of Eratosthenes you see immediately that a number is even if its units digit is even. Thus ***a whole number written in the decimal system is even if its units digit is 0, 2, 4, 6, or 8. If its units digit is not one of these the number is odd.***

We may wish to see why this is so. To do this we remember how we found the multiples of 2 when we constructed the Sieve of Eratosthenes. We started with the number 2 and added 2 again and again. The last digits repeated in a pattern, 2, 4, 6, 8, 0, 2, 4, 6, 8, 0 . . . . This would continue no matter how far we extended the table. This shows that the even numbers are those whose units digit is one of the five numbers, 0, 2, 4, 6, or 8.

If you start with 5 and added 5 again and again, you see that each of these numbers has units digit 0 or 5. Thus: ***A whole number is divisible by 5 if its units digit is 5 or 0.***

What about divisibility by 3? Can we tell by looking at the units digit of a number whether or not it is divisible by 3? The first ten muliples of 3 are:

0, 3, 6, 9, 12, 15, 18, 21, 24, 27.

Each of the possible units digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, appears in this list. On the other hand, none of 4, 7, 10, 13, 16, 19, 22, 25, 28, or 31 are divisible by 3 even though each of the possible last digits appear.

We can see, then, that we cannot tell whether a number is divisible by 3 by looking at its units digit.

But suppose we add the digits of several multiples of 3:

$12 : 1 + 2 = 3$; $15 : 1 + 5 = 6$; $18 : 1 + 8 = 9$; $21 : 2 + 1 = 3$; $24 : 2 + 4 = 6$. Making a table of the first fifteen multiples of 3 we have:

Multiple	0	3	6	9	12	15	18	21	24	27	30	33	36	39	42	45
Sum of digits	0	3	6	9	3	6	9	3	6	9	3	6	9	12	6	9

You can see a pattern here. Can you see that in each case the sum of the digits of a multiple of 3 is divisible by 3?

We may look at another number that is not divisible by 3, say 272. The sum of its digits is $2 + 7 + 2 = 11$, which is not divisible by 3. This will be true in each case where a number is not divisible by 3. In fact, if any number is divisible by 3, the sum of its digits is divisible by 3.

You may have noticed that every number in the table above that was divisible by 9 had the sum of its digits also divisible by 9. Hence, a number is divisible by 9 if the sum of its digits is divisible by 9.

Greatest Common Factor

We may consider the numbers 10 and 12. We see that both 10 and 12 are even numbers, hence they are both divisible by 2. In other words, we can say that 2 is a factor of both 10 and 12. Because 2 is a factor of both of these numbers we say it is a *common factor* of the numbers.

All whole numbers are multiples of 1. Thus 1 is a common factor of all whole numbers. Therefore, when we look for common factors of several numbers we generally look for numbers other than 1.

What factors other than 1 are common factors of 12 and 30?

$$12 = 1 \times 12 = 2 \times 6 = 3 \times 4 = 2 \times 2 \times 3$$
$$30 = 1 \times 30 = 2 \times 15 = 3 \times 10 = 5 \times 6 = 2 \times 3 \times 5$$

The set of factors of 12 are 1, 2, 3, 4, 6, and 12. The set of factors of 30 are 1, 2, 3, 5, 6, 10, 15, and 30. Hence, the common factors of 12 and 30 are 2, 3, and 6.

Do the numbers 10 and 21 have any common factor?

$$10 = 2 \cdot 5$$
$$21 = 3 \cdot 7$$

Set of factors of 10 is {1, 2, 5, 10}
Set of factors of 21 is {1, 3, 7, 21}

Hence, 10 and 21 have only one common factor, 1.

Some sets of numbers have many common factors and some sets have only 1 as a common factor.

Recognizing common factors is useful in many ways. For example, to reduce $^{1}\%_{12}$ to lowest terms you divide the numerator and denominator by the common factor 2.

We found that 12 and 30 had common factors 1, 2, 3, and 6. The largest common factor of 12 and 30 was 6. Generally this greatest common factor is more useful in mathematics than other common factors. In reducing fractions to lowest terms, you divide the numerator and denominator by their greatest common factor.

Writing the set of all factors of a number is sometimes troublesome, particularly if the number has many factors. An easier way to find the greatest common factor of several numbers is to use their prime factors.

If we want to find the greatest common factor of 72 and 90.

$72 = 2^3 \cdot 3^2$
$90 = 2 \cdot 3^2 \cdot 5$

Notice each number has 2 as a common factor and 3^2 as a common factor. Hence, the greatest common factor of 72 and 90 is $2 \cdot 3^2 = 18$.

Least Common Multiple

You already know a great deal about multiples of numbers:

1) all whole numbers are multiples of 1.
2) that the set of even numbers {0, 2, 4, . . .} are multiples of 2.
3) that the set {0, 3, 6, 9, . . .} are multiples of 3.

In fact, we can list a set of multiples of any counting number.

The number 2 is an even number and the number 3 is an odd number. At first sight these two numbers do not seem to have much in common. Yet when we look at the set of multiples of 2

{0, 2, 4, 6, 8, 10, 12 . . .}

and the set of multiples of 3

{0, 3, 6, 9, 12, 15, . . .}

we see that they do have something in common. Some of the multiples of 2 are also multiples of 3. For example 6 is a multiple of both 2 and 3. Numbers are divisible by both 2 and 3. In fact, the set of multiples of 6 are all multiples of 2 and 3.

{0, 6, 12, 18, 24, . . .}

Numbers which are multiples of more than one number are called common multiples of those numbers.

We may look at the set of multiples of 3

{0, 3, 6, 9, 12, . . .}

and the set of multiples of 4

{4, 8, 12, 16, 20, 24 . . .}

The numbers that these two sets have in common are the common multiples of 3 and 4. They are

{0, 12, 24, 36, . . .}

Common multiples are useful in arithmetic. For example, when we add two fractions like $\frac{1}{2} + \frac{1}{3}$ we must get a common denominator which is a common multiple of the two denominators.

A set of counting numbers have many common multiples. The smallest of these common multiples is called the *Least Common Multiple* of the set. The least common multiple of a set of counting numbers is defined as the smallest counting number that is a multiple of each member of the set of given numbers.

We notice that 0 is a common multiple of any set of whole numbers. In adding and subtracting fractions, 0 cannot be used as a common denominator. Why? Because of this we are interested only in the common multiples other than 0.

If we want to find the least common multiple of 12 and 20, we list the sets of multiples of each of the two numbers.

Set of multiples of 12: {0, 12, 24, 36, 48, 60, 72, 84, . . .}
Set of multiples of 20: {0, 20, 40, 60, 80, 100, . . .}

The set of common multiples of 12 and 20 is

{0, 60, 120, 180, . . .}

The smallest counting number in this set is 60. Thereafter 60 is the least common multiple of 12 and 60.

STRUCTURES FOR THE NATURAL NUMBERS

L. Clark Lay, *Professor of Mathematics, Orange County State College*

Mathematics is becoming more and more the study of form or pattern. It is concerned with the language used to describe the structure of these forms and the methods of reasoning by which we derive our conclusions. This attention to structure is today being extended to the study of arithmetic, the most elementary branch of mathematics. This is a distinct shift of emphasis, for formerly the main province of arithmetic had been computation.

The discussion here is concerned with those numbers which we represent by the numerals 1, 2, 3, 4, 5, ... These are designated, in various contexts, as the natural numbers, whole numbers, counters, or positive integers. Three structural patterns which are associated with these numbers will be considered. Two of these patterns are intrinsic—that is they are inherent in the very nature of the natural numbers. They are independent of the symbols used to represent these numbers and would still be present if we were to count #, *, @, &, ¢, . . . , instead of the usual 1, 2, 3, 4, 5, ... The third pattern is arbitrary rather than innate and has been imposed on the natural numbers or, more correctly, on the numerals by which we represent these numbers.

Let us first consider a single object, then combine this with another such, then add still another, and so on.

x	1	1
x, x	1 + 1	2
x, x, x	1 + 1 + 1	3
. . .	. . .	. . .

The "Additive" Structure

By beginning with 1, and repeatedly adding 1, any natural number may be generated. This additive structure is based on "nextness" or succession. The pattern is very simple; too simple in fact to be efficient by itself for many tasks for which these numbers can be used.

When the natural numbers are conceived in this way their properties seem so obvious and unquestionable that mathematicians themselves have only recently given them careful analysis. The following characterization of the natural numbers is essentially that given by Peano in 1899:

A. *Every natural number has a successor.* A list of natural numbers does not come to an end; there is always one more.

B. *There is a natural number, which we call 1, which is not itself the successor of any number.* Hence, we may not count #, *, @, #, . . . , or 1, 2, 3, 1, ...

C. *Two different natural numbers cannot have the same successor.* We cannot count #, *, @, *, . . . , or 1, 2, 3, 2, ...

D. *If a set of natural numbers includes 1, and if for every number in the set its successor is also included, then this set includes every natural number.* Since the set of natural numbers can never be exhibited in its entirety, we can speak of the set of "all" natural numbers only in terms of a criterion which such a set must satisfy.

These principles have had no formal use in elementary instruction. Yet the child does learn that counting begins with 1, and that it proceeds by giving the unique answer to the question "What comes next?" And in time he gains some appreciation that this series is infinite and never comes to an end.

These ideas become deeply imbedded in our consciousness and are returned to in advanced mathematical investigations when the utmost rigour is sought. A mathematical statement is considered to have a most secure foundation if it can be deduced logically from the four statements above.

We have seen how the operation of addition and a sufficient stock of 1's would enable the set of natural numbers to be constructed. What can be done with 1's in the operation of multiplication?

$1 \qquad 1$
$1 \cdot 1 \qquad 1$
$1 \cdot 1 \cdot 1 \qquad 1$
. . .

Repeated use of 1 as a factor gives only 1 as a product. Because 1 has this passive property as a multiplier, we set it aside in our task of building the natural numbers by multiplication. By introducing 2, we can obtain 2, 4, 8, 16, . . . , by multiplication alone, but a 3 will be needed as well to get 3, 6, 9, 12, ... And as is shown in the chart below, we still need 5, 7, 11, 13, . . . if we are to fill in the gaps.

		2^2	$2 \cdot 3$		2^3	3^2	$2 \cdot 5$		$2^2 \cdot 3$		$2 \cdot 7$	$3 \cdot 5$	2^4		
2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17 . . .
2		4				8								16	
	3			6			9			12					
			5					10					15		
					7							14			
									11						
											13				
															17

The "Multiplicative" Structure

The numbers 2, 3, 5, 7, 11, 13, . . . , called primes, are the fundamental building blocks for construction of the natural numbers by multiplication. The following is called The Fundamental Theorem of Arithmetic. *Every natural number, greater than 1, is either a prime, or is a unique product of primes.*

The numbers which are products of primes are called composites. They have a simple geometric interpretation. A composite number of objects can be placed in a rectangular array, with the same number in each row and the same number in each column, and with more than one row and more than one column.

We use the natural numbers to count, to tell us "how many?" Hence, we require that the numerals which represent these numbers state this information as clearly as possible. A numeral system based on either addition or multiplication alone is not the best solution. Both operations are used together with the concept of groups of standard size.

The number of days in a year (not a leap year) may be written as

365,

and this notation may be taken as an abbreviation for these forms:

Three hundred sixty five
$300 + 60 + 5$
$3(100) + 6(10) + 5(1)$
$3(10)^2 + 6(10)^1 + 5(10)^0$

The last line gives a representation as a *polynomial in powers of 10.* This polynomial is a sum of products, each product having a power of 10 as one factor. These powers of 10 are our standard groups. Easy recognition of "how many?" depends on familiarity with these groups. "365" represents a sum of products, each product being the face value of a digit times its place value.

The fundamental operations with the natural numbers consist of manipulations, in abbreviated form, of such polynomials. There has been increasing emphasis on this structure in the elementary curriculum in recent years, but without an adequate notation being used. Instead reliance has been placed on words such as, ones, tens, hundreds, and place value devices such as an abacus.

The arbitrary nature of this notation becomes apparent when we consider that powers of numbers other than ten may be used equally well for the standard groups. Thus we may again represent the number of days in a year as:

Two gross six dozen five,
or $2(144) + 6(12) + 5(1)$,
or $2(12)^2 + 6(12)^1 + 5(12)^0$,

and abbreviate this as $265_{(\text{twelve})}$, and consider this as representing, in base twelve, the number of days in a year.

Many suggestions are being made for revision of the elementary mathematics curriculum. The need for such changes rests mainly upon the assumption that a higher per cent of our school population should aspire to a much greater mastery of mathematics than in the past. Space does not allow for support of this proposition here, but evidence of its validity is already great and is growing rapidly.

It is not that the need for computational skills is growing so much, indeed we are turning an even greater share of our calculations over to machines. It is rather that the success that mathematics has enjoyed for many years in furnishing models for the physical sciences is now being extended to other disciplines such as the life, behavioral, and social sciences. To construct these models one must be conscious not only of the structure of the objects being described, but also of the mathematics being used. Hence we may expect to see more attention given to such patterns as the three we have mentioned here.

The following suggestions are made for making such studies more accessible to the elementary school pupil:

1. Revise the notations presently used to conform to standard mathematical practice.

   Replace 2×4 by $(2)(4)$ and $2(4)$ as well as $2 \cdot 4$.

   Introduce parentheses, ( ), brackets, [ ], and braces, { }, as priority symbols so that sequences of operations can be programmed. This would allow arithmetic calculations to become free of the present one-step bias. Students could then be given opportunity to read and write complete problem statements in mathematical language.

2. Introduce the operation of raising to a power.
3. Emphasize the multiplicative structure of the natural numbers. The present almost complete neglect leaves the student inadequately prepared for the rational numbers and for algebra in general.
4. Make greater use of equalities and inequalities. One can then write in sentences rather than phrases.
5. Shift the emphasis from computation to the study of form. Thus the word sum should refer to the form of an expression as well as the answer to an addition computation.

Revisions in these directions would bring the elementary school curriculum closer to the needs of today's society.

DEVELOPING MATHEMATICAL CONCEPTS THROUGH THE INDUCTIVE LEARNING PROCESS [1]

DON ALKIRE, *Professor of Mathematics, Fresno State College*

Certain mathematical concepts permeate the arithmetic curriculum from the grade school through the high school. One key concept: Finding the whole quantity when a part is known—is embodied in the following set of problems:

If 1/2 of a pie costs 30 cents,
then 2/2 of a pie costs 2×30 cents or 60 cents.

If 3/5 of Joe's weight = 45 pounds,
then 1/5 of Joe's weight = 1/3 of 45 pounds or 15 pounds;
and 5/5 of Joe's weight = 5×15 pounds or 75 pounds.

If 3/7 of Jane's marbles = 42 marbles,
then 1/7 of Jane's marbles = 1/3 of 42 marbles, or 14 marbles;
and 7/7 of Jane's marbles = 7×14 marbles or 98 marbles.

If 2 3/4 inches on a map represent 165 miles,
(i.e.) if 11/4" on a map represent 165 miles,
then 1/4" on a map = 1/11 of 165 miles or 15 miles,
then 4/4" on a map = 4×15 miles = 60 miles,
and 5" on a map = $5 \times 60 = 300$ miles.

If 13% of the weight = 1560 pounds,
then 1% of the weight = 1/13 of 1560 pounds or 120 pounds,
and 100% of the weight = 100×120 pounds or 12000 pounds.

If 37 1/2% of the bill = \$48,
(i.e.) if 3/8 of the bill = \$48,
then 1/8 of the bill = 1/3 of \$48 or \$16,
and 8/8 of the bill = $8 \times$ \$16 or \$128.

If 2/3% of the weight = 1560 pounds,
then 1/3% of the weight = 1/2 of 1560 pounds or 780 pounds,
and 3/3% of the weight = 3×780 pounds or 2340 pounds,
then 100% of the weight = 100×2340 pounds or 234,000 pounds.

[1] Based upon the presentations made by G. Don Alkire, Professor of Mathematics, Fresno State College, to study groups at the Elementary Mathematics In-service Laboratory, San Diego State College, June 19-23, 1961.

What are the ideas that emerge from working such problems? I hope that children would acquire an awareness of the relationship of a part of a thing to the whole thing. Using the first problem as an example;

If 1/2 of a pie costs 30 cents,
then 2/2 of a pie costs 2×30 cents or 60 cents.

The important ideas are that 1/2 is *half as much* as 2/2, and 2/2 is *twice as much* as 1/2. Also 2/2 of a thing is all of a thing.

Another idea I hope children would acquire is illustrated in the second problem;

If 3/5 of Joe's weight = 45 pounds,
then 1/5 of Joe's weight = 1/3 of 45 pounds or 15 pounds;
and 5/5 of Joe's weight = 5×15 pounds or 75 pounds.

An important idea is that 5/5 of a thing is *all* of a thing.

(i.e.) If 3/5 of Joe's weight is part of Joe's weight,
then 1/5 of Joe's weight is part of Joe's weight—
and is 1/3 of 3/5 of Joe's weight
and 5/5 of Joe's weight is all of Joe's weight.

An awareness of the relationship among parts of a whole thing is equally important. For example:

If 3/5 of Joe's weight = 45 pounds,
then 1/5 of Joe's weight = 1/3 of (3/5 of Joe's weight).

If 3/5 of Joe's weight is 45 pounds,
then 1/5 of Joe's weight is 1/3 of 45 pounds or 15 pounds.

If 1/5 of Joe's weight is 15 pounds,
then 5/5 of Joe's weight is 5 times 15 pounds or 75 pounds.

Mathematics is more than doing such exercises as $3\overline{)79.}$ Mathematics embodies logical argumentation. Get the if-then argumentation of mathematics under way. Consider the following problem:

If 2/3 of the class is 20 pupils, how many pupils are in the class?

Children might profit from discovering that they don't even need to know how many pupils comprise 2/3 of the class in order to begin working on the problem. The phraseology plays an important part of the learning process.

If 2/3 of the class is 20 pupils,
then 1/3 of the class must be 1/2 of 20 pupils, or 10 pupils;
if 1/3 of the class is 10 pupils,
and 3/3 of the class is all of the class,
then 3/3 of the class is 3 times 10 pupils, or 30 pupils.

The comparative factor is important in mathematics. Seek to compare like-named things. Learn to make comparisons; find the patterns; learn to make analogies; learn to make discriminations.

Mathematical principles are not acquired by working one problem. Finding the patterns may take many problems worked over a long time span.

The mathematical ideas noted in the previous discussions are applicable to similar mathematical problems used throughout the elementary school and junior high school. Such ideas are also appropriate to working problems involving proper fractions, mixed numbers, improper fractions, per cents, mixed per cents, and the like. Certain kinds of mathematical problems involving the concepts discussed may be identified in terms of difficulty and categorized by grade level as follows:

FINDING THE WHOLE QUANTITY WHEN A PART IS KNOWN

A. Lower grades

1. If 2 pencils cost 10 cents,
   then 1 pencil costs .....

2. If 1/2 of a pie costs 30 cents,
   then 2/2 of a pie cost .....

B. Middle grades

1. If 2/3 of the class is 20 pupils,
   then 1/3 of the class is 1/2 of 20 or 10 pupils,
   and 3/3 of the class is 3×10 or 30 pupils.

2. If 3/5 of Joe's marbles is 15 marbles,
   then 1/5 of Joe's marbles is 1/3 of 15 or 5 marbles,
   and 5/5 of Joe's marbles is 5×5 or 25 marbles.

C. Junior high

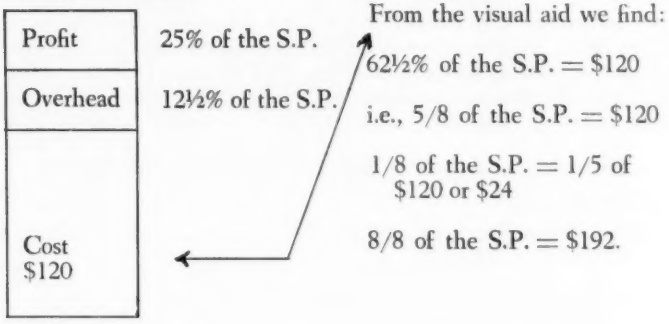
1. If 2¾ inches on a map represents 165 miles,
   (i.e.) if 11/4 inches on a map represents 165 miles,
   then 1/4 inch on a map represents 1/11 of 165 or 15 miles,
   and 4/4 inches on a map represents 4×15 or 60 miles.

2. If 13% of the weight is 1040 pounds,
   then 1% of the weight is 1/13 of 1040 or 80 pounds,
   and 100% of the weight is 100×80 or 8000 pounds.

3. If 2/3% of the county budget = $12,500,
   then 1/3% of the county budget = $6250,
   and 3/3% of the county budget = $18,750,
   therefore 100% of the county budget = $1,875,000.

4. If p pens cost q cents,
   1 pen costs 1/p of q or q/p cents,
   r pens cost $r \times q/p$ or rq/p cents.
   This type of problem is often asked in civil service examinations in various areas of work.

5. Problem: A merchant pays \$120 for a coat. Find the selling price (S.P.) so as to allow 12½% for overhead and 25% for profit, each based on the selling price.

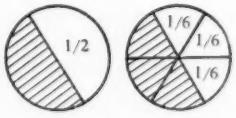

100% of the S.P.
Profit
Overhead
Cost
$120
25% of the S.P.
12½% of the S.P.
From the visual aid we find:
62½% of the S.P. = $120
i.e., 5/8 of the S.P. = $120
1/8 of the S.P. = 1/5 of $120 or $24
8/8 of the S.P. = $192.

A generalization (or rule) of great importance in the arithmetic program pertains to operations with fractions. The rule may be inferred from consideration of the following questions:

Does $1/2 = \frac{1 \times 3}{2 \times 3} = 3/6$? Does $2/3 = \frac{2 \times 3}{3 \times 3} = 6/9$?

Does $1/2 = \frac{1 \times 2}{2 \times 2} = 2/4$? Does $4/6 = \frac{4 \times 2\frac{1}{2}}{6 \times 2\frac{1}{2}} = 10/15$?

Does $2/3 = \frac{2 \times 2}{3 \times 2} = 4/6$?



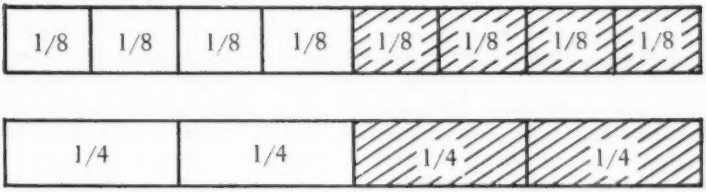
Partitioning circles in this manner may be helpful to certain children in arriving at an affirmative response to the first question. As a result of considering questions of this nature, children can infer the rule that you may multiply the numerator and denominator of a fraction by any number you choose (except zero) without changing the value of the fraction.

Another generalization (or rule) of great importance in the handling of fractions may be inferred from consideration of the following questions:

Does $4/8 = \frac{4 \div 2}{8 \div 2} = 2/4$? Does $6/9 = \frac{6 \div 3}{9 \div 3} = 2/3$?

Does $4/8 = \frac{4 \div 4}{8 \div 4} = 1/2$? Does $10/15 = \frac{10 \div 2\frac{1}{2}}{15 \div 2\frac{1}{2}} = 4/6$?

Again, using visual aids such as



certain children may be helped to perceive that an affirmative response is appropriate to the first question. As a result of considering questions of this type, children can, without prompting, infer the rule that the numerator and denominator of a fraction may be divided by any number you choose (except zero) without changing the value of the fraction.

The generalizations (or rules) mentioned above are the mathematical authority for performing certain techniques in arithmetic. We shall mention a few. Consider the following questions:

Why can a person annex zeros onto the end of decimals?

Why can one write for .12 the quantity .120?

$$.12 = \frac{12}{100} = \frac{12 \times 10}{100 \times 10} = \frac{120}{1000} = .120$$ — which was to be shown.

Why can one "move" decimal points in division, as:

$$.3\overline{)2.16} \qquad \text{(arrows under 2.1 and 6)}$$

$$.3\overline{)2.16} = \frac{2.16}{.3} = \frac{2.16 \times 10}{.3 \times 10} = \frac{21.6}{3} = 3\overline{)21.6}$$ — which was to be shown.

Why, in division of fractions, may one invert the second fraction and multiply?

$$2/5 \div 3/7 = \frac{2/5}{3/7} = \frac{2/5 \times 7/3}{3/7 \times 7/3} = \frac{2/5 \times 7/3}{1} \text{ or } 2/5 \times 7/3$$ — which was to be shown.

In each of the above types of mathematical problems, the rule which states that you may multiply the numerator and denominator of a fraction by any number you choose without changing the value of the fraction (except zero) was applied. It is the mathematical authority

or support of many of the "rules of thumb" that have evolved in arithmetic.

Children also need to know the source for the shorthand rules for working mathematical problems. The reasons which premise the rules of thumb must be made clear to the child, however, if the child is to be knowledgeable and effective at the operational level with mathematics.

Curriculum leadership personnel has the responsibility for helping teachers identify the important mathematical concepts and carry the ideas across to children. Through providing situations from which children may have sufficient experiences over a long-enough time period to take note of recurring elements in problems, children can make the generalizations, infer the principles, and develop for themselves those rules of thumb which are helpful.

Some of you at this conference have asked me to comment on a problem that often arises in the arithmetic program. What might a teacher do to demonstrate that in .12½ the ½ may apparently be changed to a 5, yielding .125.

On the chalkboard a teacher might write the following to summarize appropriate discussion?

Does .12½ = .12 ?
No, .12½ is more than .12
.12½ is .12 + some more.

Again, after further relevant questioning which culminates in a commitment by his class:

Does .12½ = .13 ?
No, .12½ is less than .13 .
Surely .12½ cannot equal .14 or .62
since it is not as large as is .13 .

A teacher willing to encourage pupils to hypothesize must be prepared to entertain the following suggestion from a pupil:

.12½ = .12 + ½.

Although this suggestion is in error, a teacher should help a student test this suggestion. The test leads to:

$$\begin{aligned} .12\tfrac{1}{2} &= .12 + \tfrac{1}{2} \\ &= .12 + .5 \\ &= .62 \end{aligned}$$

Since the class has committed itself to the acceptance of the idea that .12½ is less than .13, surely it cannot possibly be as large as .62. Hence, a pupil making the hypothesis above has earned the right by way of the personal conviction route to abandon that hypothesis.

The teacher is now ready to develop relevant experiences designed to enable pupils to furnish the following demonstration:

$.12\frac{1}{2} = .12 + 1/2 \text{ of } 1/100$
$= .12 + 1/2 \text{ of } 10/1000$ (featuring an important generalization regarding fractions)
$= .12 + 5/1000$
$= .12 + .005$
$= .120 + .005$ (annexing zero onto the end of a decimal)
$= .125$

A pupil inquiring about "changing" the ½ to 5 on .12½ wishes some demonstration with that ½. He is more likely to be satisfied with the demonstration above than with the following one, which in itself is a model of deductive logic.

Since $.12\frac{1}{2} = \frac{1}{8}$
and $.125 = \frac{1}{8}$
therefore $.12\frac{1}{2} = .125$

This is a logical demonstration. Psychologically, for some students, it may be a little wanting.

I have not made reference to anything which might be labeled as "newer mathematics." I have emphasized the need for "better teaching." I have not addressed myself to content, rather have I emphasized inductive learning, the identification and comparison of recurring elements among similar kinds of problem situations, the discovery of principles, and the thinking process characteristic of logical argument. These aspects of the arithmetic program have been emphasized by various mathematics study groups.

Children need to be able to give mathematical support for mathematical operations performed. Let us consider whether it is advisable to state the generalizations and show the shorthand rules of mathematics to children at the beginning of a teaching-learning situation. Let us consider the possible values which might accrue from learning through discovery—from inductive learning. A pupil might gain arithmetic knowledge of a more permanent nature, and as a result become more functionally competent in the subject himself, if he were furnished a teaching-learning climate which would encourage and promote such procedures as advancement of hypotheses, testing hypotheses through relevant experiences, and arriving at the end of the teaching-learning process with a generalization formulated by him in a form which he sees to hold promise of being employed to advantage.

A LOOK AT THE CURRENT POSITION OF MATHEMATICS IN THE CURRICULUM [1]

C. RICHARD PURDY, *Dean of Instruction, Alameda County State College*

These remarks are directed primarily to the programs of mathematics for the middle and upper grades.

The immediate need in preparation for changes in mathematics curriculum is for more and improved teacher education in mathematics. Teachers are called upon to make curriculum judgments and to try out new ideas; for both of these activities teachers need to be "mathematized" more thoroughly than heretofore. Both preservice and in-service programs must include more content, develop understanding of the new ideas, and provide insight for judgments on capabilities of children to learn more and different mathematical concepts.

In an effort to promote change, advance new ideas, and stimulate revision of the mathematics curriculum some mathematicians and persons with particular interest in the subject have oversold changes which should have wide-spread incorporation in the curriculum only after due time for teacher training, experimentation with content and method, and publication of new textbooks. Apparently the extremists consist of groups of persons who recommend doing away with the entire present curriculum and the institution of a totally new program and of persons who are reluctant to make any change at all because of concern for the effect of new and untried material and procedure on children. At the present time a middle course between the two extremes is necessary for changes in the mathematics curriculum and improvement of learning situations for children.

Educators should not make haste to solidify the mathematics curriculum. Not all of the material described by Dr. Willerding in the morning sessions should be taught on every grade level or to every pupil. Time will be needed to determine feasibility and grade level allocation. New-found wisdom which generates "band-wagon" curriculum changes will not necessarily improve curriculum and may actually defer development of a desirable and defensible program.

Evidence is needed of the value of many of the proposed changes. Currently, the evidence is almost entirely subjective and based on doubtful assumptions. Evaluation based on the renown of proponents, such as Dr. So-and-So, does not guarantee the value of his study for the elementary school mathematics curriculum. Report that children

[1] A summary prepared from notes taken on Dr. Purdy's presentation at the Elementary Mathematics In-Service Laboratory, June 19-23, 1961.

can accomplish certain activities and learnings is not sufficient evidence that the mathematics involved should be immediately written into the curriculum or taught to all children. Many studies and experimentations attempt justification on the basis "We had good teachers on our team," implying that because good teachers helped produce the material or found it usable or liked it that all teachers should use it. Another justification frequently heard is, "it is good mathematics." These criteria alone or in combination are not sufficient to warrant placement of the mathematics concept, skill, or method in the curriculum. However, all of these criteria—authoritative sanction, the instructional and learning possibility, the requirement of good teaching, and the quality of the mathematics—should be part of the evaluation of any project considered for curriculum incorporation.

Evidence of the achievement of children and of the value of new content and method is difficult to find at the present time because this is a period of experimentation. Furthermore, evaluation methods for determining outcomes such as intuition, relational thinking, generalizing, and the like are still undeveloped. National, state, and local groups are continuing to advance new ideas having implication for elementary school mathematics curriculum revision. They are all to be encouraged. Our current position is that they must be tried but not "bought" immediately.

With these words of precaution, it seems appropriate to mention briefly a few of the materials that have been produced to date, after which some personal recommendations for immediate implementation in the first eight grades will be made.

Among materials that are available today or that advanced notices indicate will be ready in September are the following:

The School Mathematics Study Group text material for grades seven through 12; completed material for grades four, five, and six are to be available by September 15, 1961.[2]

Scott, Foresman textbooks, *Seeing Through Arithmetic*, Grades 3-6, and seven booklets in their grade 7 Experimental Mathematics Project.[3]

Webster Publishing Company set of seven enrichment booklets, *Exploring Mathematics on Your Own.*[4]

Ginn Enrichment Program, Grades Four Through Eight, including enrichment workbooks [5] for each of grades 4, 5, and 6 and booklets on number principles [6] and "discovery" geometry.[7]

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[2] *Mathematics for Junior High School.* School Mathematics Group Study. Volume I (Part I) 1959, Volume I (Part II) 1959; Volume II (Part I) 1959, Volume II (Part II) 1960: Yale University Press, 92-A Yale Station, New Haven, Connecticut.

[3] Maurice L. Hartung, Henry Van Engen, and H. C. Trimble, *Experimental Mathematics Project.* Chicago: Scott, Foresman & Co., 1960.

[4] A. Donovan Johnson and William H. Glenn, *Exploring Mathematics On Your Own.* St. Louis: Webster Publishing Co., 1960.

[5] John L. Marks, James R. Smart, Allene Archer, William A. Brownell, Irene Sauble, and J. Fred Weaver, *Ginn Enrichment Arithmetic Program,* Grade Four, Grade Five, Grade Six. Boston: Ginn & Co., 1961.

[6] Allene Archer, *Number Principles and Patterns.* Boston: Ginn & Co., n.d.

[7] W. H. E. Bentley and E. W. Maynard Potts, *Geometry—Discovering by Drawing and Measurement,* Part I. Boston: Ginn & Co., n.d.

Holt, Rinehart, and Winston supplementary booklets for grades 4, 5, and 6.[8]

In spite of the need to proceed slowly in curriculum revision, the questions of what "new mathematics" to incorporate at the present time in the curriculum and what to use peripherally should be given consideration. Three courses of immediate action seem appropriate; revision of the regular program to include certain additional topics that work in comfortably without disruption of the present program, experimentation on a limited basis with selected classes, pupils, and teachers using new material, and supplementation of the regular program with enrichment material. Examples of resources and content for these patterns of program revision follow:

Recommendations

1. Certain additional content should be incorporated in the present hard-core program for all pupils. A few suggestions are as follows:
   a. Introduction of literal numbers into the lower grades, used in situations such as $3 \times 5 = n$, $3\frac{1}{2} \times 4\frac{1}{4} = n$, $3 + n = 8$, to provide exercise in processes. This should *not* be started with the axiomatic approach.
   b. Introduction of additional mathematics symbols to prepare children for further mathematical study. For example, $3 < 5$, $\frac{1}{2} > \frac{1}{4} \approx$ (approximately equal), and $\neq$
   c. Increased use of the number line
   d. Increased use of "If-then" form
   e. Examination of equalities as well as inequalities
   f. More mental calculation and approximation
   g. Making explicit the basic properties of our number systems of arithmetic
2. Experimentation on a limited basis with new material for selected classes, sections, or pupils from sources such as the following:
   University of Illinois
   School Mathematics Study Group
   Scott, Foresman & Co.
3. Supplementation of the regular program with enrichment content and material (in addition to the augmentation recommended in (1) above).
   Some sources for this material are the following:
   Ginn Enrichment Arithmetic Program, Ginn & Co.
   Exploring Mathematics on Your Own, Webster Publishing Co.
   Extending the New Learning Numbers, Extending the New Exploring Numbers, Extending the New Understanding Numbers: Holt, Rinehart & Winston Publishing Co.

[8] Leo J. Brueckner, Elda L. Merton, and Foster E. Grossnickle. *Extending the New Learning Numbers, Extending the New Exploring Numbers, Extending the New Understanding Numbers.* New York: Holt Rinehart & Winston Publishing Co., 1961.

A few suggestions follow:

a. Information on sets including point sets with application to non-metric geometry
b. Measures of central tendency
c. Number systems base 5, 8, 12, and 2
d. Probability
e. Historic methods for computation
f. Exponents and exponential forms for numerals
g. Generalizations on number systems such as whole numbers, counting numbers, and rational numbers
h. Number in terms of sets

Throughout the entire mathematics program more attention is being given to mental calculation, the use of more precise terminology, pupil examination of patterns and drawing conclusions, the enlargement of the concept of number, and the basic principles of mathematics. Teachers must learn more mathematics to direct this learning. We must all keep an experimental point of view, trying material with pupils, but should be slow to adopt any one program in its entirety. My prediction is that it will be from four to eight years from now before the mathematics curriculum will have stabilized.

MATHEMATICS IN THE KINDERGARTEN AND PRIMARY GRADES

MRS. LESLIE S. BEATTY, *Supervisor, Elementary Education, Chula Vista City Elementary School District*

What are the implications of the basic mathematical concepts for the education of children ranging from approximately five to nine years of age in the kindergartens and primary grades? One clearly discernible trend in curriculum change is the "mathematizing" of the traditional arithmetic. The emphasis is on helping children to understand basic mathematical concepts, to learn to think mathematically rather than exclusively on memorization of arithmetic facts and on learning to use them in the fundamental processes.

The significant elements in building mathematics curriculum for young children are (1) the selection of appropriate content; (2) the consideration of how mathematics is learned; and (3) the implementation of the mathematics program in the school.

In considering the selection of content and how mathematics is learned, a paper[1] presented by Jerome Bruner at the August, 1960, meeting of the National Council of Teachers of Mathematics merits careful review. Dr. Bruner is a noted psychologist and a writer of considerable talent. His work at Harvard University in the field of concept development is widely known. His book, *The Process of Education*,[2] is receiving widespread attention at the present time.

In briefing Dr. Bruner's ideas the impact of their vivid, thought-provoking nature may suffer. Fortunately, the presentation was published in *The Mathematics Teacher* and is available for careful reading and re-reading.

Dr. Bruner suggests three criteria for evaluating the worth of knowledge, as follows:

1. Does it give a sense of delight?
2. Does it contain within it the basis for generalization so that it leads beyond the immediate knowledge?
3. Is it useful?

With regard to the first criterion, *Does it give a sense of delight?* Dr. Bruner maintains that intellectual delight results from "the reduc-

[1] Jerome Bruner, "On Learning Mathematics," *The Mathematics Teacher*, LIII (December, 1960), 610-19.

[2] Jerome Bruner, *The Process of Education*. Cambridge, Massachusetts: Harvard University Press, 1960.

tion of surprise and complexity to predictability and simplicity"[3] with accompanying growth in the learner's sense of mastery.

Concerning the second criterion: *Does it contain within it the basis for generalization so that it leads beyond the immediate knowledge?* Dr. Bruner points out that the growth of available knowledge is so rapid that a grasp of the relatedness of knowledge is the only means by which individual knowledge can keep proportionally abreast of it. From this point of view, continuity and structure become important rather than coverage.

And finally, the third criterion: *Is it useful?* bring up the ever present question of social utility. Is it possible that education has overworked this criterion to the neglect of the first two? Or perhaps, educators may have limited the meaning of "usefulness." If mathematics as a mode of thought provides individuals with a way of bringing order out of chaos, of finding simplicity underlying complexity, is it not highly useful in this increasingly complicated life?

In considering the teaching and learning of mathematics, Dr. Bruner directs attention to discovery, intuition, translation, and readiness.

Discovery. Discovery is regarded as a process of working which is important in encouraging an active approach to learning and thinking. Discovery implies understanding and mastery; hence, it contains a reward within itself, the reward of delight in unmasking ideas.

Intuition. People generally are inclined to equate the terms "mathematics" and "rigorous thinking." Dr. Bruner points out that intuitive thinking precedes rigorous thinking. Intuitive thinking produces the hypotheses which are checked by means of analytical thinking and proof. To quote, "It is the intuitive mode . . . that produces interesting combinations of ideas before their worth is known. It . . . is only possible when the consequences of error are not overpowering or sinful . . . it is a form of activity that depends upon confidence in the worthwhileness of the process of mathematical activity rather than upon the importance of right answers at all times."

Many mathematicians recognize the importance of intuition and plausible reasoning in mathematical thought and believe that when formalism enters too early it can hamper intuitive thinking.

Robert Davis, Director of the Madison Project, expresses a similar point of view with regard to the consequences of error. The following passages from the Teacher's Manual for the first course of the project convey the idea:

> . . . nearly every student answer has some merit . . .
> We respond to every student answer as we believe scientists should . . . it is an answer, and it deserves respect. We try to avoid moral judgments, or words which suggest moral judgments. We prefer to say, "Yes, that works" (if it turns out to), rather than saying "Good" or "Right;" if an answer turns out not to work, we usually say . . . "Well, that doesn't work, do you

[3] Jerome Bruner, *The Process of Education*, *op.cit.*, p. 618.

have any other suggestions?" We want the students to feel we are partners in an experience of intellectual discovery. . . .[4]

Translation. Every teacher has searched for a simple principle or set of principles underlying a variety of instances and has then struggled to find means to make the idea clear to children. What has the teacher needed in order to perform this task? First, certainly, depth of understanding of the idea to be conveyed. Next, and just as certainly, understanding of the pupils, for the teacher must translate the idea into language or experiences which will have meaning for them. The problem of translation becomes even more difficult when the wide range of individual differences in any class is taken into account.

Dr. Bruner speaks of finding a homologue for an idea when language is inadequate for translating it. Many of the materials of instruction which are used in teaching mathematics in the primary grades are homologues, embodiments illustrative of ideas. From experiences with them, the teacher hopes that children will intuitively develop ideas which will contribute to continuity of understanding and which will translate into the language of mathematics.

Much of the instructional material available for primary children embodies mathematical ideas. All such material should be avaluated in the light of how well it conveys the ideas and whether the ideas expressed lead on to more powerful ideas.

Readiness. Dr. Bruner reports the conclusions of the Woods Hole Conference of the National Academy of Sciences that "any subject can be taught to anybody at any age in some form that is honest."[5] He does not deny the importance of maturation to readiness. However, he sees readiness as closely related to a clear understanding of what the teacher is intending to teach and his skill in translating ideas into the language and concepts of the children he is teaching. Discovery and intuitive thinking, as well as translation of ideas, are related to this concept of readiness.

At the present time in California, the Advisory Committee on Mathematics to the State Curriculum Commission is engaged in a study of the mathematics curriculum. The Committee is composed of classroom teachers, supervisors, administrators, professors of education and mathematics, and members of the State Department of Education. E. G. Begle, Director of the School Mathematics Study Group, is Consultant to the Committee.

The work of the Committee will undoubtedly influence the mathematics curriculum in California elementary and secondary schools. One of the projects in which the Committee is currently engaged is the identification of unifying "strands" of basic mathematical ideas that run through the mathematics curriculum from kindergarten through grade

[4] Robert B. Davis. *Madison Project, Workbook First Course, Teacher's Manual.* Syracuse, New York: Mathematics Department, Syracuse University, 1959.
[5] Jerome Bruner, *The Process of Education, op.cit.*

twelve. Subsequently the Committee will undertake to indicate approximately when each strand should begin and what its sequential development should be.

What is being attempted, in essence, is to seek out the underlying principles which can bring simplicity out of complexity and provide structure and continuity.

The Committee has currently identified the following unifying strands of basic mathematical ideas:

1. Number, numeration, and operations on numbers
2. Number systems
3. Applications and problem solving
4. Sets
5. Geometry
6. Measurement
7. Mathematical sentences, equations
8. Logic and definition
9. Function
10. Probability

Dr. Bruner's article and the work of the State Advisory Committee recalls many specific experiences with young children and their many opportunities to use measurement and number and to work with a variety of materials. One group of first grade children made scrapbooks; their latest one completed at the end of the year was titled "Sets." The children collected many pictures for this and classified the sets by their number properties. The children, guided by a skillful, knowledgeable teacher expressed in mathematical sentences ideas gained from a variety of experiences. The children worked at different levels; for example, Dana writes $1 + 1 = 2$, and $4 > 3$, while Jimmy writes $98 + 2 = 100$ and $3 + 5 < 4 + 5$.

This first grade teacher made a chart of generalizations at which children had arrived. Before they made these observations, the children had worked with counters, pictures, and perception cards. They had learned to use the number line. They had developed a tens chart. They had used an add-subtract slide rule the group had made. The teacher recorded what the children had found out to enhance their delight in discovery. The record was composed of the following items:

WHAT WE FOUND OUT ABOUT NUMBERS

0 and 1 cannot be taken apart.—Ronald
When we add 1 it makes a larger number.—Gayla
When we add 1 it makes the next higher number.—David S.
Counting by tens is like counting by ones, taking away the zeros.—David J.
When we count by twos we skip a number.—Linda

When we count by twos starting with two the numbers are even numbers.—Pamela
When we count by fives we skip over four numbers.—David J.
The digit at the end of the number is five or zero.—Carol
When we take away it leaves a lesser number.—Diane
When we take away 1 we get the number that comes before it.—David S.

This first grade is evidence that many of the basic concepts can be translated into the language and experiences of children in the primary grades and that the methods of teaching of which Dr. Bruner speaks do work.

A modern program of mathematics in the primary grades will be implemented as teachers have opportunity to learn the basic mathematical concepts, to observe skilled teachers guiding the learning of children, and to experiment with basic ideas being revealed in the important research going on in the many studies of mathematics now in progress throughout the nation.

THE PLACE OF DEMONSTRATION TEACHING IN IMPROVING INSTRUCTION IN ELEMENTARY SCHOOL MATHEMATICS

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Demonstration teaching is a technique for in-service education which has many values. The opportunity to observe a well-prepared lesson in mathematics often helps teachers set new standards for themselves and gain new insights into the personalities and needs of children.

Asking a teacher to demonstrate for other teachers is a well-accepted method of giving recognition to exceptional skill and accomplishment. A consultant or supervisor may often take the opportunity afforded by an approaching demonstration to help the teacher raise his level of performance by planning with him or suggesting various ways to present the lesson. In an atmosphere of "we are going to meet the public," suggestions can be made and ideas injected which the supervisor and teacher might be reluctant to discuss in the usual run of daily affairs.

Demonstrations by a general consultant or by a consultant in a special field also have great value. The material presented in this paper is based on experiences in demonstration teaching done by a consultant in elementary school mathematics.

A need for demonstration teaching should first be established. Teachers are usually eager to observe a demonstration because many have had little preparation in teaching the aspects of mathematics now being emphasized in the elementary schools and, therefore, feel inadequate and insecure. Some teachers doubt that the methods advocated will be successful with children, and a classroom demonstration may convince them of the value of a new approach.

Careful planning must precede the selection of the problem to be developed in the demonstration lesson. The administrator and the general consultant often gather the teachers' requests for assistance. These requests form a solid basis for planning. Teachers may examine test results, make an item analysis, and request help on specific areas of need. The administrator or the general consultant may suggest topics based on test results or on their close observation of classroom activities. The mathematics consultant will know where trouble is likely to arise and may provide a demonstration to prepare teachers to meet the difficulty; or the consultant may decide to introduce a technique or teaching

aid regarding which no questions have been raised by teachers. For instance the use of the number line was demonstrated to teachers of both primary and upper grades. The response has been enthusiastic.

For the sake of economy in the use of time, care is given to selecting a problem which will afford opportunity to draw upon several concepts which may need clarification; for example, regrouping in subtraction may include work on place value, and the importance of expressing a value in many ways, as well as on the principles of setting up subtraction problems.

The classroom in which the demonstration is to be held should be carefully selected. The person demonstrating should never work with a class unless the teacher is entirely willing to have the group used in the demonstration. Some teachers have difficulty in observing their pupils objectively. These teachers may be so concerned that the class does well that they cannot profit from the demonstration.

Demonstrations may be held for the teachers within a building, for the teachers within a district, or a county. Problems of freeing the teachers to observe the demonstrations have been met by having the county or the district provide a floating substitute, by combining classes, or by declaring a board holiday for in-service education. When classes are combined, care should be taken to see that the children whose teachers are freed to observe have a worth-while experience. Administrators and general consultants should attend the demonstrations as they have responsibility to help the teachers evaluate and use the ideas presented. Teachers usually prefer to observe a demonstration at the grade level they are teaching, and many request a series of demonstrations on the work of one specific grade level. Probably meeting this need should have precedence, but values in understanding children and the curriculum accrue when teachers observe at grade levels other than the one at which they are teaching.

Whenever possible, an orientation meeting is held before the demonstration. Appropriate information regarding the situation and the purpose of the lesson is presented. The group may be asked to give special attention to the variety of expressions used or to the specific respects in which the topic to be dealt with fits into the larger arithmetic program.

The meeting following the demonstration is of great importance. The teachers are encouraged to ask questions which may be answered by the demonstration teacher, the regular teacher, the general supervisor or the administrator. Discussion may become general, and many ideas can thus be shared.

Ample time should be scheduled for demonstration activities. One suggested schedule follows:

Orientation meeting	8:00 a.m. to 8:50 a.m.
Demonstration lesson	9:10 a. m to 9:50 a.m.
Follow-up discussion	9:50 a.m. to 10:30 a.m.

(The length is adjusted to the grade level.)

In some situations, two or three demonstrations are held within a building or a district within the same day, using the services of the floating substitute. In such cases, the follow-up discussions are held after school.

Teachers appreciate the help they receive from observing demonstrations. When they observe the work of their own pupils many seem to gain new insight into both mathematical and personality needs. One teacher said that the demonstration was like holding up a mirror to himself. When well prepared for the demonstration, teachers show fine professional attitudes toward the experience.

Two anecdotes presented by a mathematics consultant illustrate this point:

> Following the demonstration I was having an individual conference with the teacher whose class I had taught. The teacher wondered how a certain child was able to do what he had done. In an exchange of pleasantries, I told the teacher I had an advantage because I did not know the boy could not do the work. Then the teacher asked how I knew how to help the boy reason his way through the process. His question afforded the opportunity to stress the importance of examining a process to determine its component elements, and the importance of planning thoroughly in order to be able to help a child. The teacher expressed this deduction, "In other words you have to know your subject."
>
> Several months after another demonstration a teacher rushed up to me to say that a miracle had happened to Jane. According to the teacher, Jane was a girl who had great ability, but who was not succeeding in mathematics. I had caught the gleam in her eye while teaching the class, and had asked her a difficult question which she answered correctly. The teacher said that finding she could answer this question had been the turning point in the girl's work for the year. What he did not see is that *he* had learned that day that Jane could reason in mathematics. It was no doubt the turning point in the teacher's attitude toward her that had changed the situation for Jane.

Teachers recognize other values in observing demonstrations, as indicated by the list that follows:

1. Observing helps teachers see the components of a concept and the sequence followed in arriving at a generalization.
2. Observing a demonstration class helps teachers clarify their own understandings of mathematical operations.
3. Observation reveals gaps in teaching and points up the needs of individuals and groups.
4. Teachers can learn new techniques and methods and the use of new materials through observing a demonstration lesson.
5. Teachers come to see and to accept the need for thorough planning.

From the point of view of the arithmetic consultant demonstration teaching serves many purposes. It affords opportunities for the following:

1. To develop mathematical understandings with teachers. Teaching teachers through working with children protects the feelings of teachers who may feel insecure in teaching arithmetic
2. To illustrate techniques of developing concepts with children, for example: (1) the need for determining the components of concepts in advance of teaching; (2) the importance of a logical sequence in the presentation of the lesson; and (3) the importance of much oral discussion before any independent written work is assigned
3. To demonstrate the use of well selected equipment and help teachers establish high standards for its selection.
4. To discover points of difficulty experienced by children
5. To plan for further work in the light of the needs which have been revealed

Keeping in close touch with children enables the arithmetic consultant to be practical rather than theoretical in work suggested for children.

The general consultants and the administrators also recognize the significant values in demonstrations. They report that the opportunity to observe special lessons in mathematics contributes to their feeling of adequacy in the field and helps them plan the important follow-up work. The specific nature of the additional work needed by their pupils is often discovered during the demonstration lesson. The task of seeing that the needs are dealt with falls upon those who work with the teachers regularly.

The section which follows lists examples of demonstration lessons which have been used successfully.

1. Lessons based on the results of diagnostic tests

   Teachers analyze the tests and send in the results with requests for a meeting or for demonstration teaching followed by a meeting.

   The tests may reveal the need for such seemingly simple work as is outlined by the following questions and instructions:

   Which is larger, twelve hundred or one thousand?
   What number comes next after 9999?
   Write 4050 using words.
   Write eight hundred six using digits.
   Subtract 38 from 85.

   The chief need in connection with the use of diagnostic tests is the need for organization. Teachers tend to run through the items in the order in which they appear in the test and they welcome suggestions for grouping items which involve similar concepts.

2. Lessons based on such content of the state textbooks in arithmetic as that which follows:
   a. the concept of the commutative law of addition and multiplication
   b. place value
   c. the distributive law of multiplication
   d. the use of simple scales
   e. reading and constructing graphs
3. Lessons planned to supplement the state textbooks not present in content that follows:
   a. the number line
   b. the formulating of simple equations and the use of literal numbers
   c. the use of the number sentence and the vocabulary of mathematics
4. Lessons based on concepts not included in the state textbooks that present the content that follows:
   a. bases other than 10
   b. division as a series of equal subtractions
   c. the meaning of the reciprocal in the division of fractions.

Carefully planned demonstration lessons can improve and enrich instruction in mathematics. One of the purposes of this instruction is to help children think logically and precisely. Children may become better thinkers if they move out of the fields of the concrete and the specific into generalizations based on understanding. A group of special activities recommended for deepening the understandings in mathematics is in the list that follows:

1. Provide pupils with problems drawn from their social studies
2. Provide many problems which include irrelevant data
3. Provide many problems which lack sufficient data
4. Provide many problems without numbers
5. Have pupils construct for class use problems of the types mentioned above
6. Help pupils relate mathematics to their problems as consumers—using newspaper advertising, mail order catalogues, labels, and cartons
   a. discover the cost of carrying charges
   b. discover the error in some so-called "economy buys"
   c. examine labels for critical features
7. Reduce the amount of abstract, practice work required to that needed by the pupil
8. Encourage pupils to do much of their computing "in their heads"
9. Help pupils discover several ways to work the same problem

10. Use supplementary arithmetic textbooks designed for the grade level of the pupils
11. Provide interesting books rich in mathematical concepts as a category under which pupils may choose books to read in a program of self selection
12. With pupils who have achieved proficiency in a process, substitute activities which call for the application of mathematical principles already learned
13. Have pupils apply their knowledge by constructing materials, such as open-end abaci or Napier's rods
14. Have pupils make an inventory of the use of standard measures in the home, in the school, and in the community and make a traveling kit which may circulate to other classrooms
15. Help pupils relate arithmetic to science activities—ratio as used in constructing pulleys; measuring and graphing as related to animal feeding experiments and to principles of soil conservation
16. Have pupils apply their knowledge in constructing graphs—bar, picture, line, or circle, depending on the level of their ability to deal with the processes involved. Raw data should be used.
17. Have pupils construct statistical maps using data related to their social studies units and to current happenings. Raw data should be used, as one of the values of this experience is to help children select and organize the facts for presentation.

Many parents are greatly interested in how mathematics is now being taught in the elementary schools. In order to take advantage of their interest and readiness to help, and also to preclude conflicts in the children's thinking, meetings may be held to acquaint them with procedures in mathematics which may be new or different.

Much of the material presented at this conference has been properly concerned with learning how to use the so-called "new" mathematics. Participants have also received help on improving the presentation of the basic operations—the mathematics which has been around for a long time. Both are important.

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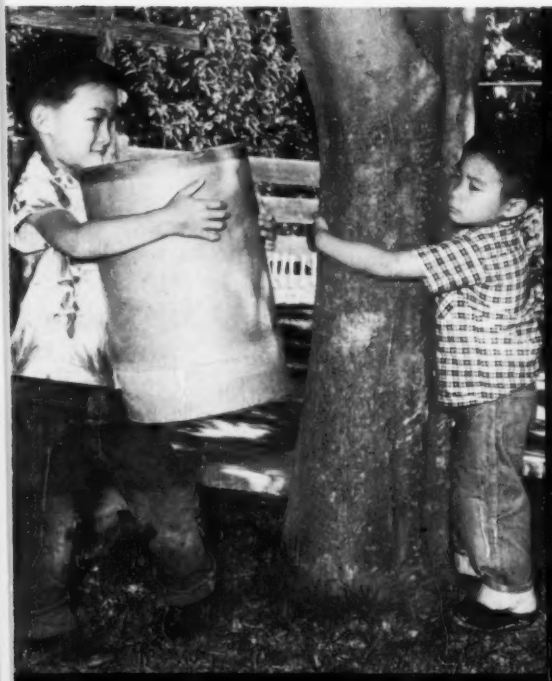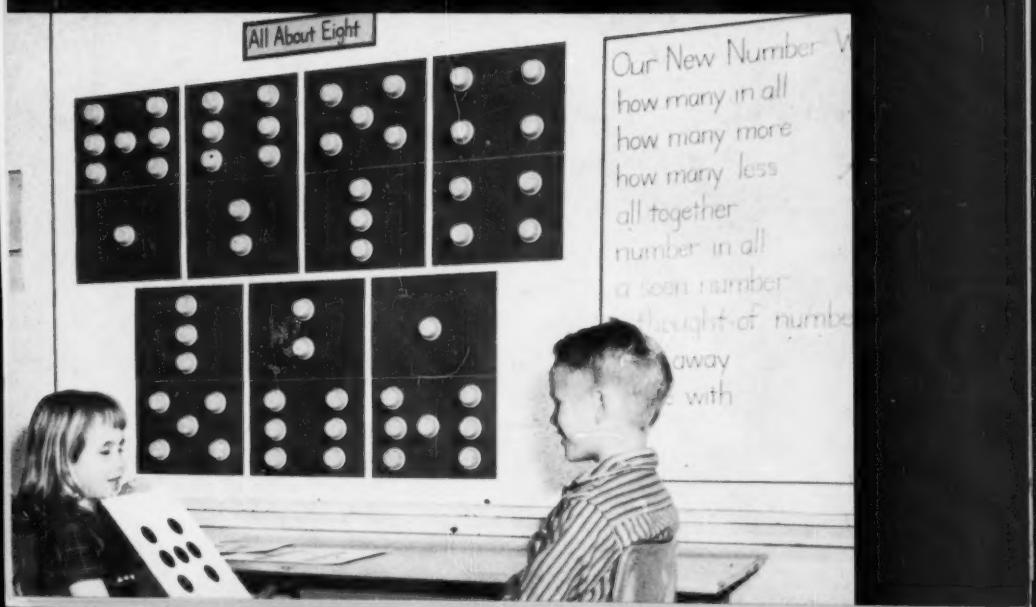
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